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Tutorial

on

Analytical Treatment of Some Fundamental Topics in Antenna Theory.

by

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List of Topics.

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LIST OF TOPICS :

- 1. Brief introduction to notation and some preliminary concepts including charge as a fundamental source of EM fields.
  - (a) Current flow viewed as net motion of charges. Development of complete expressions for the exact EM fields of general current distributions.
  - (b) Various approximations for the corresponding fields of antennas in their reactive, Fresnel and Fraunhofer regions, respectively.
  - (c) Complex EM power conservation theorem and its use in developing a general expression for antenna impedance. Use of the antenna impedance in describing an equivalent circuit for a transmitting antenna.
- 2. Development of EM Reciprocity and Reaction Theorems for Antennas in the frequency domain.
  - (a) Field/Lorentz form of reciprocity theorem.
  - (b) Development of circuit form of the reciprocity theorem for a pair of coupled antennas. Concept of mutual impedance/admittance. A two port network for representing a pair formed by a transmitting and a receiving antenna system.
  - (c) Generalized reciprocity/reaction theorem in mixed circuit-field form leading to specific field based expressions for calculating the mutual impedance/admittance between a pair of antennas.
    Examples involving coupled dipoles, as well as coupled slots in a ground plane are discussed.
- 3. Additional applications of generalized reciprocity/reaction theorems.
  - (a) Reaction (generalized reciprocity) theorem for calculating the voltage induced in a receiving antenna from a distant/far zone transmitting antenna.
    A Thevenin/Norton circuit for a receiving antenna.
  - (b) Reaction theorem for calculating the far zone radiation pattern of a patch antenna without the use of a complicated Sommerfeld integral form of the microstrip Green's function.
- 4. Slotted rectangular waveguide array analysis.
  - (a) Resonant (narrow band) broad-wall slotted waveguide array theory for generating a broadside beam.
  - (b) Non resonant broad-wall slotted waveguide array as a leaky wave antenna for frequency scanned beam applications. Forward and backward beams.

References

[1] P. H. Pathak and R. J. Burkholder, Electromagnetic Radiation, Scattering, and Diffraction (IEEE Press Series on Electromagnetic Wave Theory), IEEE Press, WILEY, N.J., 2022.

[2] A. D. Yaghjian, "An overview of near field antenna measurements," IEEE Trans. AP-34, Vol. 1, pp. 30-45, 1986.

[3] R. F. Harrington*, Time-Harmonic Electromagnetic Fields, McGraw-Hill book company, New York, 1961.

[4] R. E. Collin, Antennas and Radiowave Propagation, McGraw-Hill book company, New York, 1985.

(Additional General Reference on Antenna Theory:
C. A. Balanis, Antenna Theory, Wiley Interscience, N.Y., Third Edition, 2005)

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* R. F. Harrington, Time-Harmonic Electromagnetic Fields, IEEE Press series on Electromagnetic Wave Theory. IEEE Press, WILEY, N.J., 2001.

Useful papers on reaction theorems:

- V. H. Rumsey, "Reaction concept in electromagnetic theory", Physical Review, vol. 94, pp. 1483-1491, June 15, 1954.

- J. H. Richmond, "A Reaction Theorem and Its Application to Antenna Impedance Calculations," IRE Transactions on Antennas and Propagation, pp. 515-520, November 1961.

(Also see [3] for applications of reaction theorems).

Some Fundamental Relations in Antenna Theory:

- Electric charge constitutes the basic source of electromagnetic (EM) fields.
- Motion of charges (net drift) constitutes a flow of current.
- Charge is conserved.
- Rate of decrease of charge density (charge/vol.) in a region must therefore be simultaneously accompanied by a flow of electric current flow out of that region. [CONTINUITY EQUATION]
- Time varying currents radiate EM fields.

- ⇒ TIME HARMONIC SCALAR/VECTOR QUANTITIES:

$$\mathcal{A}(\bar{r},t) = Re[A(\bar{r},\omega)]\, e^{+j\omega t} \quad ; \quad j=\sqrt{-1}\,.$$

$\mathcal{A}(\bar{r},t)$: SPACE ($\bar{r}$), time (t) $\qquad$ $A(\bar{r},\omega)$: SPACE ($\bar{r}$), angular frequency (ω)

$\omega = 2\pi f$; f = frequency (Hz).

A can be $\bar{E}, \bar{D}, \bar{B}, \bar{H}, \bar{J}$ or ρ_e

$\mathcal{A}$ can be $\bar{\mathcal{E}}, \bar{\mathcal{D}}, \bar{\mathcal{B}}, \bar{\mathcal{H}}, \bar{\mathcal{J}}$ or ϱ_e.

(e.g.,: $\bar{\mathcal{E}} = Re\,\bar{E}\,e^{j\omega t}$; $\varrho_e = Re\,\rho_e\,e^{j\omega t}$.)

Maxwell's Equations for isotropic homog. media:

- **ARBITRARY TIME DEPENDENCE:**

MAXWELL EQNS.

$$\boxed{\nabla\times\bar{\mathcal{E}} = -\frac{\partial\bar{\mathcal{B}}}{\partial t}^{?} \; ; \; \nabla\times\bar{\mathcal{H}} = \frac{\partial\bar{\mathcal{D}}}{\partial t}^{?} + \bar{\mathcal{J}} \; ; \; \nabla\cdot\bar{\mathcal{D}} = \varrho_e \; ; \; \nabla\cdot\bar{\mathcal{B}} = 0 .}$$

$$\bar{\mathcal{J}} = \underbrace{\bar{\mathcal{J}}_i}_{\text{impressed}} + {}^{?}\underbrace{\bar{\mathcal{J}}_c}_{\text{conduction}} \; ; \quad \bar{\mathcal{J}}_c = \int_{-\infty}^{t} \sigma_e(\bar{r}, t-\tau)\,\bar{\mathcal{E}}(\bar{r}, t)\,d\tau$$

$$\bar{\mathcal{D}} = \epsilon_0\bar{\mathcal{E}} + \bar{\mathcal{P}} \; ; \quad \bar{\mathcal{P}} = \int_{-\infty}^{t} \chi_e(\bar{r}, t-\tau)\,\bar{\mathcal{E}}(\bar{r}, t)\,d\tau$$

$$\bar{\mathcal{B}} = \mu_0(\bar{\mathcal{H}} + \bar{\mathcal{M}}) \; ; \quad \bar{\mathcal{M}} = \int_{-\infty}^{t} \chi_m(\bar{r}, t-\tau)\,\bar{\mathcal{H}}(\bar{r}, t)\,d\tau$$

CONSTITUTIVE RELATIONS

$$\varrho_e = \varrho_{ei} + \varrho_{ec}$$

CONTINUITY EQN. $\Rightarrow$ $\nabla\cdot\bar{\mathcal{J}} = -\dfrac{\partial\varrho_e}{\partial t} \rightarrow \nabla\cdot\bar{\mathcal{J}}_i = -\dfrac{\partial\varrho_{ei}}{\partial t}$ and $\nabla\cdot\bar{\mathcal{J}}_c = -\dfrac{\partial\varrho_{ec}}{\partial t}$

$\Rightarrow$ *Maxwell's Div. eqns. are not indep.!!*

NOTE:
$$\bar{\mathcal{E}}(\bar{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}_e(\bar{r}, \omega)\,e^{j\omega t}\,d\omega \; , \text{ etc.}$$

$$\bar{f}_e(\bar{r}, \omega) = \int_{-\infty}^{\infty} \bar{\mathcal{E}}(\bar{r}, t)\,e^{-j\omega t}\,dt \; , \text{ etc.}$$

$\Rightarrow$ FOR A HOMOG. MEDIUM, σ_e, χ_e, and χ_m are <u>not</u> a function of $\bar{r}$.

- **TIME HARMONIC CASE:**

$$f_e(\bar{r}, \omega') = 2\pi\,\bar{E}(\bar{r}, \omega')\left[\frac{\delta(\omega'-\omega) + \delta(\omega'+\omega)}{2}\right]$$

$$\bar{\mathcal{E}}(\bar{r}, t) = \int_{-\infty}^{\infty} \bar{E}(\bar{r}, \omega')\left[\frac{\delta(\omega'-\omega) + \delta(\omega'+\omega)}{2}\right] e^{j\omega' t}\,d\omega'$$

$$= \frac{1}{2}\left[\bar{E}(\bar{r}, \omega)e^{j\omega t} + \bar{E}^{*}(\bar{r}, \omega)e^{-j\omega t}\right]$$

$$\bar{\mathcal{E}}(\bar{r}, t) = \mathrm{Re}\,\bar{E}(\bar{r}, \omega)\,e^{j\omega t} .$$

MAXWELL EQNS.

$$\boxed{\nabla\times\bar{E} = -j\omega\bar{B} \; ; \; \nabla\times\bar{H} = \bar{J} + j\omega\bar{D} \; ; \; \nabla\cdot\bar{D} = \rho_e \; ; \; \nabla\cdot\bar{B} = 0 .}$$

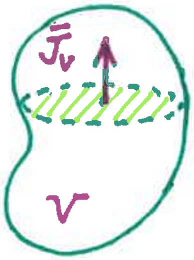
$$\bar{J} = \bar{J}_i + \bar{J}_c \; ; \; \bar{J}_c = \sigma_e\bar{E} \; ; \; \bar{D} = \epsilon_0\bar{E} + \chi_e\bar{E} \; ; \; \bar{B} = (\mu_0 + \chi_m)\bar{H}$$

CONSTITUTIVE RELATIONS

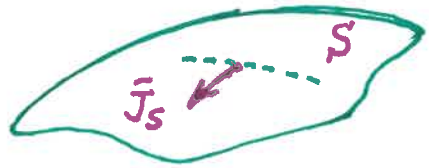
$$\bar{D} = \epsilon\bar{E} \; ; \; \bar{B} = \mu\bar{H} \; ; \quad \nabla\cdot\bar{J} = -j\omega\rho_e$$

CONT. EQN.

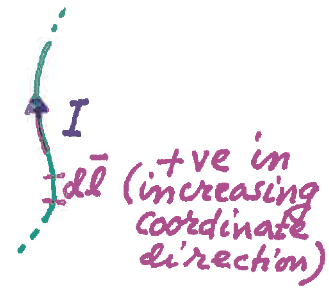
Volume, Surface, and Line Current Distributions



(a) Volume current density $\bar{J}_v$ (amps/m²)



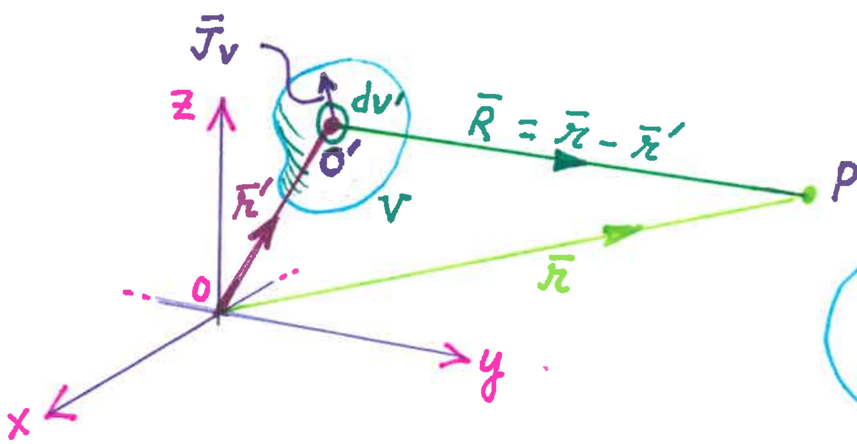
(b) Surface current density $\bar{J}_s$ (amps/m)



(c) Line current I (amps)

$$d\bar{p}_e = \begin{pmatrix} \bar{J}_v \, dv \\ \bar{J}_s \, ds \\ I \, d\bar{\ell} \end{pmatrix} \rightarrow \text{infinitesimal electric current moment.}$$

For a POINT electric current source, $\bar{J}_v = \bar{p}_e \delta(\bar{r} - \bar{r}')$, at $\bar{r}'$, and $d\bar{p}_e = \bar{p}_e \delta(\bar{r}' - \bar{r}'') dv'' \longrightarrow d\bar{p}_e = d\bar{p}_e(\bar{r}')$.



$\bar{J}_v \in V$ produces $(\bar{E}, \bar{H})$.

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \quad ; \quad \nabla \times \bar{H} = \bar{J}_v + j\omega\epsilon \bar{E}$$

$\therefore$ $\nabla \times \nabla \times \bar{E} = -j\omega\mu(\bar{J}_v + j\omega\epsilon\bar{E})$, with $\nabla\times\nabla\times = \nabla(\nabla\cdot) - \nabla^2$

or

$$\boxed{[\nabla^2 + k^2]\bar{E} = j\omega\mu\left[\bar{J}_v + \frac{1}{k^2}\nabla(\nabla\cdot\bar{J}_v)\right]} \; ; \; k^2 \equiv \omega^2\mu\epsilon = \frac{\omega^2}{c^2}$$

Also $k = \omega/c = 2\pi(f/c) = \frac{2\pi}{\lambda}$; λ = wavelength.

EM Fields Produced by a Source $\bar{J}$ in an Unbounded Homog. Isotropic Medium.

The electric field $\bar{E}(\bar{r})$ due to a source $\bar{J}(\bar{r}')$ in V has been shown to satisfy the pde:

$$[\nabla^2 + k^2]\bar{E} = [\bar{\bar{I}} + \tfrac{1}{k^2}\nabla\nabla]\cdot(j\omega\mu\bar{J}).$$

The above is the same as:

$$\boxed{\nabla\times\nabla\times\bar{E} - k^2\bar{E} = -j\omega\mu\bar{J}.}$$ arbitrary source

(Note: $\bar{\bar{I}}$ is an identity dyad, so $\bar{\bar{I}}\cdot\bar{a} = \bar{a} = \bar{a}\cdot\bar{\bar{I}}$).

One can introduce a "<u>dyadic Green's function</u>", $\bar{\bar{G}}_o$ for unbounded regions which is the response to a <u>spatially "impulsive" source</u> (Dirac Delta Fcn.):

$$[\nabla^2 + k^2]\bar{\bar{G}}_o = [\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}]\delta(\bar{r}-\bar{r}')$$

OR

$$\boxed{\nabla\times\nabla\times\bar{\bar{G}}_o - k^2\bar{\bar{G}}_o = -\bar{\bar{I}}\delta(\bar{r}-\bar{r}')}$$ delta fcn. source

(<u>since $\nabla\times\nabla\times = \nabla(\nabla\cdot) - \nabla^2$</u>).

The above suggests that $\bar{\bar{G}}_o$ can be expressed as

$$\bar{\bar{G}}_o(\bar{r}|\bar{r}') = -[\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}]G_o(\bar{r}|\bar{r}').$$

where

$$(\nabla^2 + k^2)G_o(\bar{r}|\bar{r}') = -\delta(\bar{r}-\bar{r}').$$

Thus,

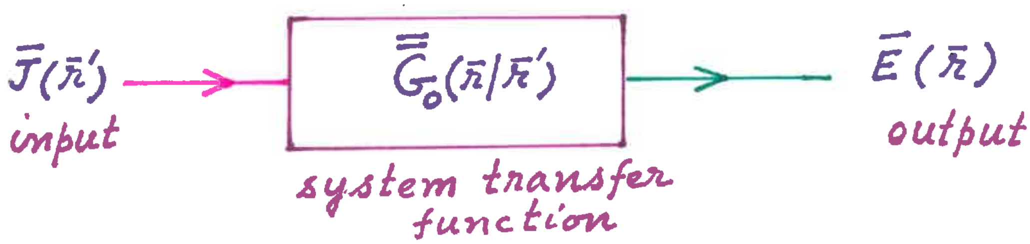
$$G_o(\bar{r}|\bar{r}') = \frac{e^{-jkR}}{4\pi R}; \quad R = |\bar{R}| = |\bar{r}-\bar{r}'|.$$

(<u>subject to the "outgoing" wave condition</u>).

The solution for $\bar{E}$ is thus given by [1]

$$\bar{E}(\bar{r}) = j\omega\mu \int_V \bar{\bar{G}}_0(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')\,dv'$$

OUTPUT — CONVOLUTION INTEGRAL — IMPULSE RESPONSE (TRANSFER FUNCTION) — INPUT



FROM CKT THEORY ⇒ $y(t) = \int_{-\infty}^{t} h(t-\tau)\, f(\tau)\, d\tau$

output — CONVOLUTION INTEGRAL — folded and shifted impulse response (transfer fcn.) — input

$$\bar{\bar{G}}_0(\bar{r}|\bar{r}') = -\left[\bar{\bar{I}} + \frac{\nabla\nabla}{k^2}\right] \frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}.$$

? **CHECK**: Is the equation at the top a solution to

$$\nabla\times\nabla\times\bar{E}(\bar{r}) - k^2\bar{E}(\bar{r}) = -j\omega\mu\bar{J}(\bar{r}) \quad ??$$

$$j\omega\mu\left[\nabla\times\nabla\times\int_V \bar{\bar{G}}_0(\bar{r}|\bar{r}')\cdot\bar{J}(\bar{r}')dv' - k^2\int_V \bar{\bar{G}}_0(\bar{r}|\bar{r}')\cdot\bar{J}(\bar{r}')dv'\right] = -j\omega\mu\int_V \bar{J}(\bar{r}')\delta(\bar{r}-\bar{r}')dv'$$

(since $\int_V \bar{J}(\bar{r}')\cdot\bar{\bar{I}}\delta(\bar{r}-\bar{r}')dv' = \int_V \bar{J}(\bar{r}')\delta(\bar{r}-\bar{r}')dv' = \bar{J}(\bar{r})$)

⇒ For $\bar{r}\neq\bar{r}'$, the preceeding relation yields the following

$$\nabla\times\nabla\times\bar{\bar{G}}_0 - k^2\bar{\bar{G}}_0 = -\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}'),$$

which is correct, as indicated previously.

Dot Product between a Dyadic and a Vector

Vector : $\bar{A} = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z$

Dyadic : $\bar{\bar{T}} = \hat{x}\hat{x}\, T_{xx} + \hat{x}\hat{y}\, T_{xy} + \hat{x}\hat{z}\, T_{xz} +$
$+ \hat{y}\hat{x}\, T_{yx} + \hat{y}\hat{y}\, T_{yy} + \hat{y}\hat{z}\, T_{yz} +$
$+ \hat{z}\hat{x}\, T_{zx} + \hat{z}\hat{y}\, T_{zy} + \hat{z}\hat{z}\, T_{zz}$.

$$\bar{\bar{T}} \cdot \bar{A} = T_{xx} A_x \hat{x} (\hat{x} \cdot \hat{x}) + T_{yx} A_x \hat{y} (\hat{x} \cdot \hat{x}) + T_{zx} A_x \hat{z} (\hat{x} \cdot \hat{x})$$
$$+ T_{xy} A_y \hat{x} (\hat{y} \cdot \hat{y}) + T_{yy} A_y \hat{y} (\hat{y} \cdot \hat{y}) + T_{zy} A_y \hat{z} (\hat{y} \cdot \hat{y})$$
$$+ T_{xz} A_z \hat{x} (\hat{z} \cdot \hat{z}) + T_{yz} A_z \hat{y}\, \hat{z} \cdot \hat{z} + T_{zz} A_z \hat{z} (\hat{z} \cdot \hat{z})$$

$$\bar{C} \equiv \bar{\bar{T}} \cdot \bar{A} = \hat{x} (T_{xx} A_x + T_{xy} A_y + T_{xz} A_z)$$
$$+ \hat{y} (T_{yx} A_x + T_{yy} A_y + T_{yz} A_z)$$
$$+ \hat{z} (T_{zx} A_x + T_{zy} A_y + T_{zz} A_z)$$

or

$$[\bar{C}] = [\bar{\bar{T}}] \cdot [\bar{A}] = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} .$$

$\bar{D} \equiv \bar{A} \cdot \bar{\bar{T}} = \bar{\bar{T}}^{t} \cdot \bar{A}$; superscript "t" → transpose.

Check:

$$[\vec{A}]\cdot[\overline{\overline{T}}] = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

OR

$$\bar{A}\cdot\overline{\overline{T}} = \hat{x}\,(A_x T_{xx} + A_y T_{yx} + A_z T_{zx})$$
$$+ \hat{y}\,(A_x T_{xy} + A_y T_{yy} + A_z T_{zy})$$
$$+ \hat{z}\,(A_z T_{zx} + A_y T_{zy} + A_z T_{zz})$$

From above, it is clear that a rearrangement yields:

$$[\bar{A}]\cdot[\overline{\overline{T}}] = \begin{bmatrix} T_{xx} & T_{yx} & T_{zx} \\ T_{xy} & T_{yy} & T_{zy} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$\therefore$

$$\bar{A}\cdot\overline{\overline{T}} = \overline{\overline{T}}^{\,t}\cdot\bar{A}$$

TRANSPOSE

$$\bar{E}(\bar{r}) = j\omega\mu \int_V \bar{\bar{G}}_o(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')\, dv'$$

$$\bar{E}(\bar{r}) = \hat{x}\, E_x(\bar{r}) + \hat{y}\, E_y(\bar{r}) + \hat{z}\, E_z(\bar{r})$$

$$\bar{J}(\bar{r}') = \hat{x}\, J_x(\bar{r}') + \hat{y}\, J_y(\bar{r}') + \hat{z}\, J_z(\bar{r}')$$

$$\bar{\bar{G}}_o(\bar{r}|\bar{r}') = \hat{x}\hat{x}\, G_{oxx} + \hat{x}\hat{y}\, G_{oxy} + \hat{x}\hat{z}\, G_{oxz} + \hat{y}\hat{x}\, G_{oyx} + \hat{y}\hat{y}\, G_{oyy} + \hat{y}\hat{z}\, G_{oyz} + \hat{z}\hat{x}\, G_{ozx} + \hat{z}\hat{y}\, G_{ozy} + \hat{z}\hat{z}\, G_{ozz}.$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = j\omega\mu \int_V \begin{bmatrix} G_{oxx} & G_{oxy} & G_{oxz} \\ G_{oyx} & G_{oyy} & G_{oyz} \\ G_{ozx} & G_{ozy} & G_{ozz} \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} dv'$$

$\Rightarrow$ (e.g. $G_{ozz} = -\left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right]\frac{e^{-jkR}}{4\pi R}$.)

More generally, one can write [1]:

$$\bar{E}(\bar{r}) = j\omega\mu \int \bar{\bar{G}}_o(\bar{r}|\bar{r}') \cdot d\bar{p}_e(\bar{r}') \; ; \; d\bar{p}_e(\bar{r}') = \begin{pmatrix} \bar{J}_v\, dv\, ; \\ \bar{J}_s\, ds\, ; \\ I\, d\bar{l}\, . \end{pmatrix}$$

or

$\Rightarrow$
$$\bar{E}(\bar{r}) = \int d\bar{E}(\bar{r}) \; ; \; d\bar{E}(\bar{r}) = j\omega\mu\, \bar{\bar{G}}_o(\bar{r}|\bar{r}') \cdot d\bar{p}_e(\bar{r}').$$

From [1] (see pgs. 165-167) a coordinate free representation is:

$\Rightarrow$
$$d\bar{E}(\bar{r}) = \frac{jkZ}{4\pi}\left[\left(\hat{R}\times\hat{R}\times d\bar{p}_e(0')\right)\left(1 - \frac{j}{kR} - \frac{1}{(kR)^2}\right) - \left(2\hat{R}\hat{R}\cdot d\bar{p}_e(0')\right)\left(\frac{j}{kR} + \frac{1}{(kR)^2}\right)\right]\frac{e^{-jkR}}{R}.$$

$\Rightarrow$
$$d\bar{H}(\bar{r}) = \frac{-jk}{4\pi}\left[\left(\hat{R}\times d\bar{p}_e(0')\right)\left(1 - \frac{j}{kR}\right)\right]\frac{e^{-jkR}}{R}.$$

(Note: $Z = Y^{-1} \equiv \sqrt{\frac{\mu}{\epsilon}}$)

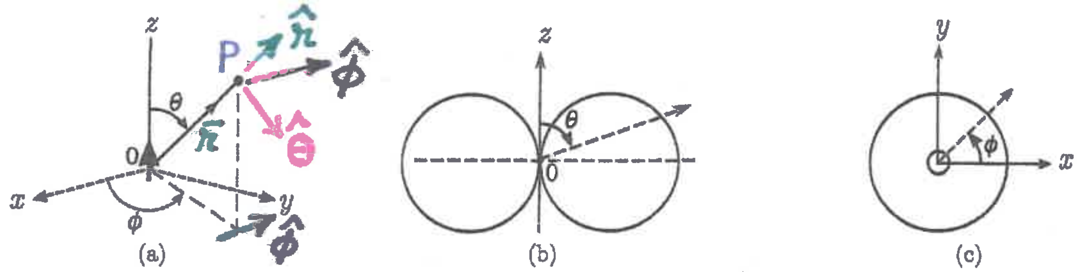


Figure A $\hat{z}$-directed electric current moment of strength dP_e at the origin, 0, of a rectangular co-ordinate system. Also the far zone radiation pattern of this $\hat{z}$-directed current moment is shown in two principal planes. (a) Geometry, (b) pattern in E-plane or ϕ =CONSTANT, and (c) pattern in H-plane or $\theta = \pi/2$.

If $\bar{r}' = 0$; $\bar{R} = \bar{r} - \bar{r}' = \bar{r} \rightarrow R = r$.

- Let $d\bar{p}_e(\bar{r}')\,(= d\bar{p}_e(0'))$ be positioned so $\bar{r}' \equiv 0$
- Also, let $d\bar{p}_e(0) \equiv \hat{z}\, dp_e(0)$.

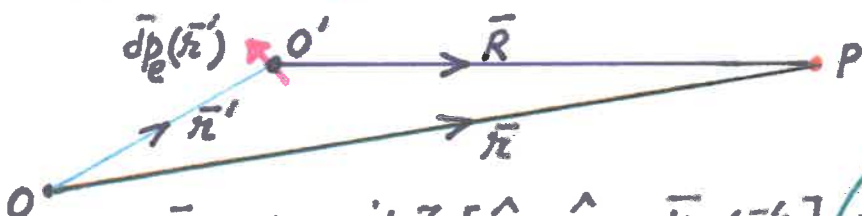
Thus,

- $$d\bar{E}(\bar{r}) = d\bar{E}(P) = \frac{jkZ}{4\pi} dp_e(0)\left[(\hat{\theta}\sin\theta)\left(1 - \frac{j}{kr} - \frac{1}{(kr)^2}\right) - (2\hat{r}\cos\theta)\left(\frac{j}{kr} + \frac{1}{(kr)^2}\right)\right]\frac{e^{-jkr}}{r}.$$

- $$d\bar{H}(\bar{r}) = d\bar{H}(P) = \frac{jk}{4\pi} dp_e(0)\left[(\hat{\phi}\sin\theta)\left(1 - \frac{j}{kr}\right)\right]\frac{e^{-jkr}}{r}.$$

NOTE : $\hat{r}\times\hat{r}\times\hat{z} = -\hat{r}\times\hat{\phi}\sin\theta = \hat{\theta}\sin\theta$; $\hat{r}\cdot\hat{z} = \cos\theta$.

Co-ordinate free representation for any $d\bar{p}_e(\bar{r}')$ and $kR \gg 1$



$$d\bar{E}(\bar{r}) \approx \frac{jkZ}{4\pi}\left[\hat{R}\times\hat{R}\times d\bar{p}_e(\bar{r}')\right]\frac{e^{-jkR}}{R} \quad ; \ kR \gg 1$$

SPHERICAL WAVE

$$d\bar{H}(\bar{r}) \approx \frac{jk}{4\pi}\left[\hat{R}\times d\bar{p}_e(\bar{r}')\right]\frac{e^{-jkR}}{R} \quad ; \ kR \gg 1.$$

Thus: $d\bar{E} = -Z\hat{R}\times d\bar{H}$; $d\bar{H} = Y\hat{R}\times d\bar{E}$ (with $Y = Z^{-1}$)

The above indicates a "LOCAL" plane wave behaviour.

Antenna Near and Far Fields

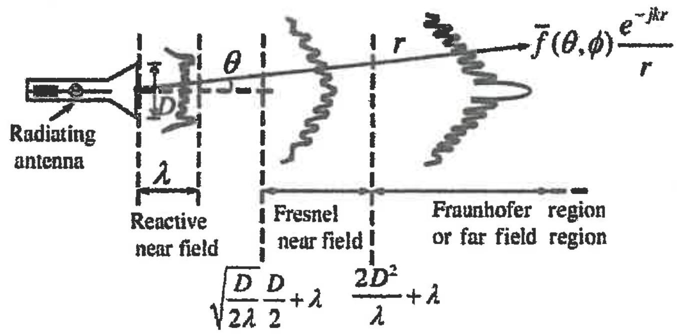


Figure : Field distribution produced by a radiating antenna. The distribution is plotted in a plane transverse to the antenna axis (boresight) and at increasing distances from the antenna, namely at λ, $\sqrt{\frac{D}{2\lambda}\cdot\frac{D}{2}+\lambda}$ and $\frac{2D^2}{\lambda}+\lambda$, respectively. [2].

$$\bar{E}(\bar{r}) = \int d\bar{E}(\bar{r}) = \frac{jkZ}{4\pi}\int\Big[\big(\hat{R}\times\hat{R}\times d\bar{p}_e(\bar{r}')\big)\Big(1-\frac{j}{kR}-\frac{1}{(kR)^2}\Big) - \big(2\hat{R}\hat{R}\cdot d\bar{p}_e(\bar{r}')\big)\Big(\frac{j}{kR}+\frac{1}{(kR)^2}\Big)\Big]\frac{e^{-jkR}}{R};$$

$$\bar{H}(\bar{r}) = \int d\bar{H}(\bar{r}) = \frac{-jk}{4\pi}\int\Big[\big(\hat{R}\times d\bar{p}_e(\bar{r}')\big)\Big(1-\frac{j}{kR}\Big)\Big]\frac{e^{-jkR}}{R}.$$

- **Reactive Near Field Region** $(0 < r < \lambda)$
  Fields dominated by $\left(\frac{1}{kR}\right)^3$ terms. Also $\bar{E}$ and $\bar{H}$ are almost 90° out of phase. Thus reactive field region (antenna stored energy region).

- **Radiating Near Field Region** $\left(\lambda < r < \frac{2D^2}{\lambda}+\lambda\right)$
  Amplitude terms retained to $O\left[\left(\frac{1}{kR}\right)^2\right]$.
  e^{-jkR} can be approximated by:

$$e^{-jkR} = e^{-jk|\bar{r}-\bar{r}'|} = e^{-jk\sqrt{(\bar{r}-\bar{r}')\cdot(\bar{r}-\bar{r}')}} = e^{-jkr\sqrt{1+\left(\frac{r'}{r}\right)^2-2\frac{\hat{r}\cdot\bar{r}'}{r}}}$$

$$e^{-jkR} \approx e^{-jk\left(r-\hat{r}\cdot\bar{r}'+\frac{r'^2}{2r}\left[1-(\hat{r}\cdot\hat{r}')^2\right]\right)}.$$

A part of the radiating near field region where $\sqrt{\frac{D}{2\lambda}}\cdot\frac{D}{2}+\lambda < r < \frac{2D^2}{\lambda}+\lambda$, is referred to as the Fresnel Near Field region; here, usually the following approximation may be employed:

$$\bar{E}(\bar{r}) \approx \frac{jkZ}{4\pi}\int\left(\left[\hat{R}\times\hat{R}\times d\bar{p}_e(\bar{r}')\right]\left(1-\frac{j}{kR}\right)+\left[2j\hat{R}\hat{R}\cdot d\bar{p}_e(\bar{r}')\right]\left(\frac{1}{kR}\right)\right)\cdot \frac{1}{R}e^{-jk\left[r-\hat{r}\cdot\bar{r}'+\frac{r'^2}{2r}\left(1-[\hat{r}\cdot\hat{r}']^2\right)\right]};$$

$$\bar{H}(\bar{r}) \approx \frac{-jk}{4\pi}\int\left(\left[\hat{R}\times d\bar{p}_e(\bar{r}')\right]\left(1-\frac{j}{kR}\right)\right)\cdot\frac{1}{R}e^{-jk\left[r-\hat{r}\cdot\bar{r}'+\frac{r'^2}{2r}\left(1-[\hat{r}\cdot\hat{r}']^2\right)\right]}.$$

As seen above, the Fresnel region is one for which the "quadratic" approximation for the dominant phase term e^{-jkR} becomes valid.

- Radiating Far Field Region $\left(r > \frac{2D^2}{\lambda}+\lambda\right)$

The radiating far field region is also referred to as the Fraunhofer region. In this region the "linear" approximation for the dominant phase term e^{-jkR} becomes valid. Also, only $\frac{1}{kR}$ terms need to be retained in amplitude. In this FAR ZONE approximation:

$$|\bar{r}| >> |\bar{r}'| \rightarrow r >> r' \quad (\text{with } r'_{max} \approx D).$$

The $\bar{E}$ and $\bar{H}$ fields are in phase within the FAR ZONE.

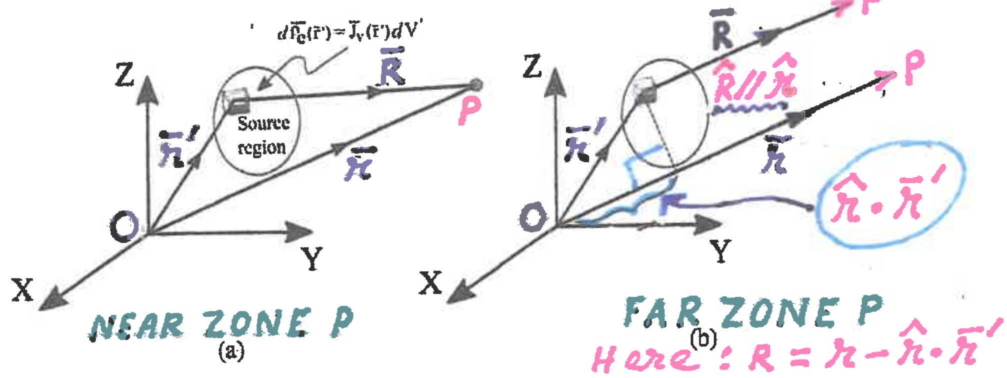


Figure: (a) Near Zone ; (b) Far Zone (PARALLEL RAY APPROX.)

In the far field (or FAR ZONE) the following approximations become valid:

$$\hat{R} \approx \hat{r} \quad ; \quad e^{-jkR} \approx e^{-jkr} \, e^{+jk\hat{r}\cdot\bar{r}'}$$

$$\bar{E}(\bar{r}) \approx \frac{jkZ}{4\pi}\left[\hat{r}\times\hat{r}\times\int d\bar{p}_e(\bar{r}')e^{jk\hat{r}\cdot\bar{r}'}\right]\frac{e^{-jkr}}{r} \equiv \frac{jkZ}{4\pi}\bar{P}_E(\hat{r})\frac{e^{-jkr}}{r}$$

$$\bar{H}(\bar{r}) \approx \frac{-jk}{4\pi}\left[\hat{r}\times\int d\bar{p}_e(\bar{r}')e^{jk\hat{r}\cdot\bar{r}'}\right]\frac{e^{-jkr}}{r} \equiv -\frac{jk}{4\pi}\bar{P}_H(\hat{r})\frac{e^{-jkr}}{r}$$

LOCAL PLANE WAVE → $\hat{r}\cdot\bar{E} = 0 = \hat{r}\cdot\bar{H}$; $\bar{E} = -Z\hat{r}\times\bar{H}$ (or $\bar{H} = Y\hat{r}\times\bar{E}$)

$$\bar{P}_E(\hat{r}) = \hat{r}\times\hat{r}\times\int d\bar{p}_e e^{jk\hat{r}\cdot\bar{r}'} \; ; \; \bar{P}_H(\hat{r}) = \hat{r}\times\int d\bar{p}_e e^{jk\hat{r}\cdot\bar{r}'}$$

Note : All contributions $d\bar{p}_e e^{jk\hat{r}\cdot\bar{r}'}$ summed ($\int d\bar{p}_e e^{jk\hat{r}\cdot\bar{r}'}$) together according to the vectorial behaviour of $d\bar{p}_e(\bar{r}')$ and phase behaviour according to $e^{jk\hat{r}\cdot\bar{r}'}$ within the entire source region gives rise to the antenna vector RADIATION PATTERN ($\bar{P}_E$ for $\bar{E}$; $\bar{P}_H$ for $\bar{H}$).

The vector radiation pattern changes only with $\hat{r}$; it is NOT dependent on the far zone distance r, provided $r > \frac{2D^2}{\lambda} + \lambda$ (the extra term λ is to accomodate antennas whose dimensions are $\leq \frac{\lambda}{2}$).

EM Power Conservation Theorem and Antenna Impedance.

$\bar{\mathcal{P}}(\bar{r},t) \equiv \bar{\mathcal{E}}(\bar{r},t) \times \bar{\mathcal{H}}(\bar{r},t)$, is the INSTANTANEOUS POWER DENSITY $\left(\frac{\text{volts}}{m} \cdot \frac{\text{amps}}{m} = \frac{\text{watts}}{m^2}\right)$.

$$\nabla \cdot \bar{\mathcal{P}} = \nabla \cdot (\bar{\mathcal{E}} \times \bar{\mathcal{H}}) = \bar{\mathcal{H}} \cdot \nabla \times \bar{\mathcal{E}} - \bar{\mathcal{E}} \cdot \nabla \times \bar{\mathcal{H}}$$

Maxwell's eqns.: $\nabla \times \bar{\mathcal{E}} = -\frac{\partial \bar{\mathcal{B}}}{\partial t}$; $\nabla \times \bar{\mathcal{H}} = \bar{\mathcal{J}} + \frac{\partial \bar{\mathcal{D}}}{\partial t}$.

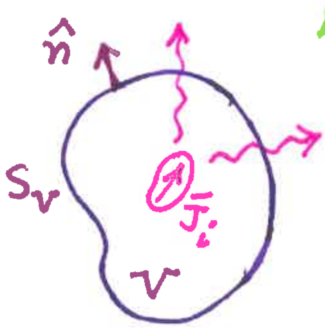
$$\Rightarrow \bar{\mathcal{J}} = \bar{\mathcal{J}}_i + \bar{\mathcal{J}}_c$$

$$-\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_i = \nabla \cdot \bar{\mathcal{P}} + \bar{\mathcal{E}} \cdot \frac{\partial \bar{\mathcal{D}}}{\partial t} + \bar{\mathcal{H}} \cdot \frac{\partial \bar{\mathcal{B}}}{\partial t}$$

Divergence makes physical sense only when evaluated under an integral sign. Thus integrating the above over a volume containing $\bar{\mathcal{J}}_i$ yields:

$$\underbrace{\int_V [-\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_i]\, dv}_{\text{Power generated by } \bar{\mathcal{J}}_i \text{ in } V} = \underbrace{\oint_{S_V} \bar{\mathcal{P}} \cdot \hat{n}\, ds}_{\text{Power leaving } S_V} + \underbrace{\int_V \left[\bar{\mathcal{E}} \cdot \frac{\partial \bar{\mathcal{D}}}{\partial t} + \bar{\mathcal{H}} \cdot \frac{\partial \bar{\mathcal{B}}}{\partial t}\right] dv}_{\text{Power stored in } V \text{ together with power lost in } V \text{ due to dielectric/magnetic losses.}} + \underbrace{\int_V \bar{\mathcal{E}} \cdot \bar{\mathcal{J}}_c\, dv}_{\text{Power lost in } V \text{ due to resistive effects (Joule heating)}}$$

Note: $\int_V \nabla \cdot \bar{\mathcal{P}}\, dv = \oint_{S_V} \bar{\mathcal{P}} \cdot \hat{n}\, ds$ (Divergence Theorem).



The above integral involving 5 terms is a statement of CONSERVATION OF POWER.

For the <u>time harmonic</u> ($e^{+j\omega t}$) case :

TIME AVERAGE POWER DENSITY = $\bar{P}_{avg} = \frac{1}{T}\int_0^T \bar{\mathcal{P}}\,dt$; $T = \frac{2\pi}{\omega}$

$$\bar{P}_{avg} = \frac{1}{T}\int_0^T \bar{\mathcal{E}} \times \bar{\mathcal{H}}\,dt = \frac{1}{T}\int_0^T (\mathrm{Re}\,\bar{E}e^{j\omega t}) \times (\mathrm{Re}\,\bar{H}e^{j\omega t})\,dt$$

$$\bar{P}_{avg} = \langle \bar{\mathcal{P}} \rangle = \frac{1}{2}\mathrm{Re}\,\bar{E} \times \bar{H}^*$$

(since $\mathrm{Re}\,\bar{a}e^{j\omega t} = \dfrac{\bar{a}e^{j\omega t} + \bar{a}^* e^{-j\omega t}}{2}$, etc.)

Let $\bar{P}_{avg} \equiv \mathrm{Re}\,\bar{P}$; $\bar{P} \equiv \frac{1}{2}\bar{E} \times \bar{H}^*$

Next one evaluates $\nabla \cdot \bar{P}$:

$$\nabla \cdot \bar{P} = \frac{1}{2}\nabla \cdot (\bar{E} \times \bar{H}^*) = \frac{1}{2}\bar{H}^* \cdot \nabla \times \bar{E} - \frac{1}{2}\bar{E} \cdot \nabla \times \bar{H}^*$$

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \qquad \nabla \times \bar{H} = \bar{J}_i + \bar{J}_c + j\omega\epsilon\bar{E}$$

Note $\bar{\mathcal{J}}_{i,c} = \mathrm{Re}\,\bar{J}_{i,c}e^{j\omega t}$, etc. ..., $\int_V \nabla \cdot \bar{P}\,dv = \oint_{S_V} \bar{P} \cdot \hat{n}\,ds$

Thus,

$$-\frac{1}{2}\int_V \bar{J}_i^* \cdot \bar{E}\,dv = \oint_{S_V} \bar{P} \cdot \hat{n}\,ds + \frac{1}{2}\omega\int_V [\mu''|\bar{H}|^2 + \epsilon''|\bar{E}|^2]\,dv +$$
$$+ \frac{1}{2}\int_V \sigma|\bar{E}|^2 dv + j\frac{\omega}{2}\int_V [\mu'|\bar{H}|^2 - \epsilon'|\bar{E}|^2]\,dv ,$$

with

$\bar{J}_c = \sigma\bar{E}$; $\epsilon = \epsilon' - j\epsilon''$; $\mu = \mu' - j\mu''$. (valid for $e^{+j\omega t}$).

NOTE: (ϵ'', μ'') are positive here.

The above expression involving integral states that:

(COMPLEX power generated by $\bar{J}_i$ in V) = (COMPLEX power leaving S_V) + (power lost in V due to dielectric and magnetic damping forces) + (resistive loss in V → Joule heating) + (reactive power stored in V).

∴ "POWER IS CONSERVED".

⇨ (Note : $\bar{E} \cdot \bar{E}^* = |\bar{E}|^2$; $\bar{H} \cdot \bar{H}^* = |\bar{H}|^2$).

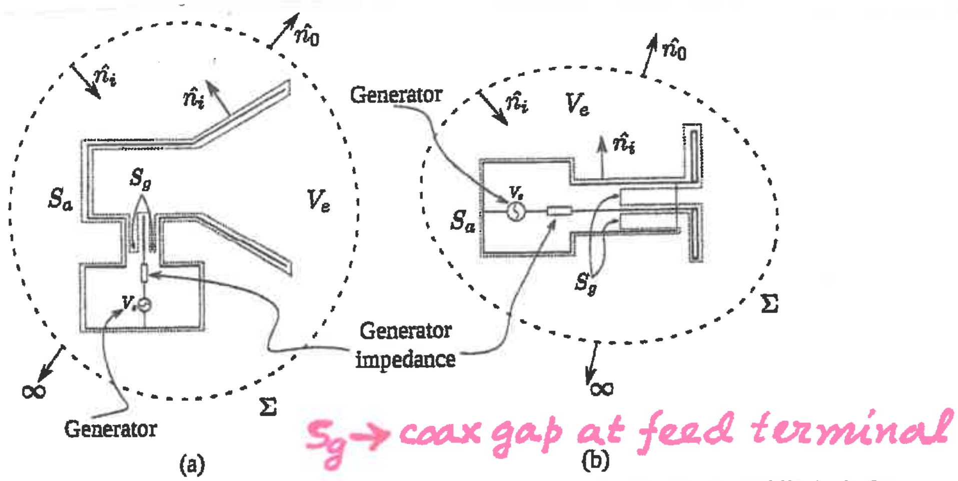


Figure : Representation of an antenna geometry with well defined feed terminals. While (a) is designated as a horn type and (b) is designated as a dipole type, these geometries are hypothetical and either one can be used to represent a generic antenna. The generic antenna surface is designated by S_a, whereas the feed gap S_g is used to define the set of feed terminals. Here the feed is assumed to be a coaxial cable.

The expression for conservation of power can be utilized to develop the concept of ANTENNA IMPEDANCE.

Let the mathematical surface S tightly encapsulate any generic antenna geometry. Let the external volume V_e be enclosed by $S+\Sigma$, where Σ is a closed surface at ∞.

From the previous expression for power conservation applied to V_e bounded by $S+\Sigma$, one obtains

$$\underbrace{\cancel{-\frac{1}{2}\int_{V_e} \bar{J}_i^* \cdot \bar{E}\, dv}}_{0} = \oint_{S+\Sigma} \bar{P}\cdot \hat{n}_o\, ds + P_d + jP_s \text{ , since } \bar{J}_i \notin V_e .$$

$$\left(\bar{P} = \frac{1}{2}\bar{E}\times\bar{H}^*\right)$$

$$P_d \equiv \int_{V_e} \left[\frac{1}{2}\sigma|\bar{E}|^2 + \frac{\omega}{2}\left(\mu''|\bar{H}|^2 + \epsilon''|\bar{E}|^2\right)\right] dv$$

$$P_s \equiv \int_{V_e} \frac{\omega}{2}\left[\mu'|\bar{H}|^2 - \epsilon'|\bar{E}|^2\right] dv$$

Also, $\oint_{S+\Sigma} \bar{P}\cdot\hat{n}\,ds$ can be expressed with $S = S_a + S_g$ as

$$\oint_{S+\Sigma} \bar{P}\cdot\hat{n}_o\,ds = -\int_{S_a}\bar{P}\cdot\hat{n}_i\,ds - \int_{S_g}\bar{P}\cdot\hat{n}_i\,ds + \int_{\Sigma}\bar{P}\cdot\hat{n}_o\,ds$$

S_a tightly covers the antenna structure.
S_g is the annular coax feed gap at feed terminals.

Combining the above terms for $\oint_{S+\Sigma}\bar{P}\cdot\hat{n}_o\,ds$, P_d and P_s yields,

$$\oint_{\Sigma}\bar{P}\cdot\hat{n}_o\,ds + P_d + jP_s = \int_{S_a}\bar{P}\cdot\hat{n}_i\,ds + \int_{S_g}\bar{P}\cdot\hat{n}_i\,ds$$

$$= \underbrace{-P_{da}}_{\text{(power lost in antenna surface)}} + \underbrace{\tfrac{1}{2}V_t I_t^*}_{\text{(complex power at antenna input } (S_g))}$$

Here V_t and I_t are the circuit voltage and current at the well defined feed terminals (S_g), where a SINGLE CO-AX mode (TEM) exists. In particular,

$$\frac{1}{2}\int_{S_g}\bar{E}\times\bar{H}^*\cdot\hat{n}_i\,ds = \frac{1}{2}\int_{S_g}(V_t\,\bar{e})\times(I_t\,\bar{h})^*\cdot\hat{n}_i\,ds\;.$$

The TEM modal fields in the coax are normalized such that $\int_{S_g}\bar{e}\times\bar{h}^*\cdot\hat{n}_i\,ds = 1$. Thus, one obtains

$$\frac{1}{2}V_t I_t^* = \oint_{\Sigma}\frac{1}{2}\bar{E}\times\bar{H}^*\cdot\hat{n}_o\,ds + P_{da} + P_d + jP_s \rightarrow \text{POWER CONSERVATION}$$

$$\frac{1}{2}\mathrm{Re}\,V_t I_t^* = \frac{1}{2}\mathrm{Re}\oint_{\Sigma}\bar{E}\times\bar{H}^*\cdot\hat{n}_o\,ds + (\mathrm{Re}\,P_{da} + P_d)$$

$$\frac{1}{2}\mathrm{Im}\,V_t I_t^* = \frac{1}{2}\mathrm{Im}\oint_{\Sigma}\bar{E}\times\bar{H}^*\cdot\hat{n}_o\,ds + (\mathrm{Im}\,P_{da} + P_s)$$

$$Z_a = \begin{matrix}\text{ANTENNA}\\ \text{IMPEDANCE}\\ \text{AT } S_g \text{(INPUT)}\end{matrix} \equiv \frac{V_t}{I_t} = R_a + jX_a \Rightarrow V_t = Z_a I_t$$

$\therefore$

$$\frac{1}{2}|I_t|^2 R_a = \frac{1}{2} Re \oint_{\Sigma} \bar{E}\times\bar{H}^* \cdot \hat{n}_o \, ds + (Re\, P_{da} + P_d)$$

(DOMINANT)

$$\frac{1}{2}|I_t|^2 X_a = \frac{1}{2} Im \oint_{\Sigma} \bar{E}\times\bar{H}^* \cdot \hat{n}_o \, ds + (Im\, P_{da} + P_s).$$

(DOMINANT)

$$\frac{1}{2}|I_t|^2 R_r \equiv \frac{1}{2} Re \oint_{\Sigma\to\infty} \bar{E}\times\bar{H}^* \cdot \hat{n}_o \, ds \; ; \quad R_r \equiv \text{RADIATION RESISTANCE}$$

$\therefore$

$$\frac{1}{2}|I_t|^2 R_a = \frac{1}{2}|I_t|^2 R_r + (Re\, P_{da} + P_d)$$

Typically P_d is very small. Also, P_{da} can be made small by choosing low loss material for the antenna structure. A large R_r implies large power radiated by the antenna. Additionally $\frac{1}{2} Im \oint_{\Sigma\to\infty} \bar{E}\times\bar{H}^* \cdot \hat{n}_o \, ds \to 0$

since the reactive power resides mostly close to the antenna.

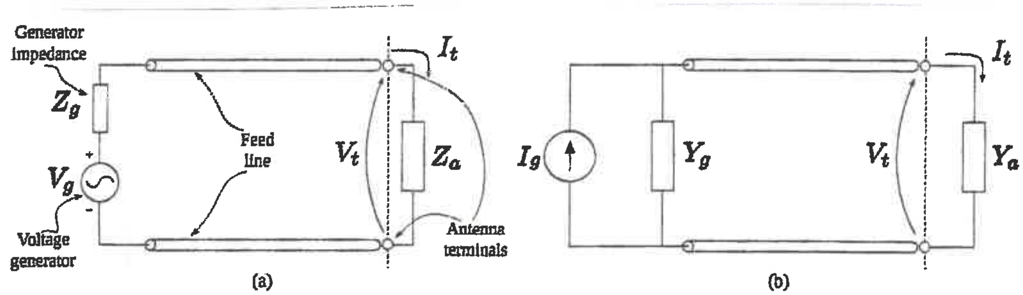


Figure An equivalent circuit for an antenna under transmitting conditions. A generator feeding an antenna via a transmission (or coax) line. The antenna appears as a load to the line at the antenna terminals. (a) Voltage generator and (b) current generator. (see [1]).

Duality Theorem

It is useful to introduce the concept of FICTICIOUS magnetic currents and charges. In reality, $\nabla \cdot \bar{B} = 0$, because magnetic charges (POLES) always appear in pairs. However, ficticious isolated magnetic poles are introduced for various reasons, namely they introduce symmetry into Maxwell's eqns., and are useful as equivalent sources in a variety of electromagnetic equivalence theorems which are useful in the formulation of EM problems. The duality theorem is a consequence of symmetry in Maxwell's equations.

$$\nabla \times \bar{H} = \bar{J} + j\omega \bar{D} \quad ; \quad \nabla \times \bar{E} = -\bar{M} - j\omega \bar{B} .$$
$$\nabla \cdot \bar{D} = \rho \quad ; \quad \nabla \cdot \bar{B} = \rho_m$$

Also, $\nabla \cdot \bar{J} = -j\omega\rho$ and $\nabla \cdot \bar{M} = -j\omega \rho_m$.

$\bar{E}$	$\bar{H}$
$\bar{H}$	$-\bar{E}$
$\bar{J}$	$\bar{M}$
$\bar{M}$	$-\bar{J}$
ρ	ρ_m
ρ_m	$-\rho$
$\bar{D}$	$\bar{B}$
$\bar{B}$	$-\bar{D}$
ϵ	μ
μ	ϵ
Z	Y
Y	Z

DUAL PAIRS

- Replacing quantities within the LEFT COLUMN in Maxwell's equations with the corresponding quantities within the RIGHT COLUMN shows Maxwell's eqns. remain UNCHANGED.

- Solution to a given problem can therefore directly furnish the solution to its DUAL PROBLEM without EXTRA effort.

EM Reciprocity and Reaction Theorems useful for Antenna Applications [1]

⇨ : Some uses of reciprocity/reaction principle:

- It is useful in offering clues for meaningful approximations in the solution of EM problems.
- It lends some physical insights into the measurement of antennas as reactions (-EM- OBSERVABLES)
- It is used in formulating expressions for antenna mutual coupling as reactions
- It is useful in relating EM antenna and scattering problems
- It is useful in obtaining EM equivalence theorems

⋮

- Let a source pair $(\bar{J}_a, \bar{M}_a)$ generate the fields $\bar{E}_a$ and $\bar{H}_a$ in a volume V (enclosed by a closed surface S_V) and at a frequency "f" $(=\frac{\omega}{2\pi})$.
- Let a source pair $(\bar{J}_b, \bar{M}_b)$ generate $\bar{E}_b$ and $\bar{H}_b$ in the SAME VOLUME V (bounded by S_V) and at the SAME "f" but with the sources $(\bar{J}_a, \bar{M}_a)$ TURNED OFF.

"IN GENERAL", THE ENVIRONMENT IN V MAY BE DIFFERENT.

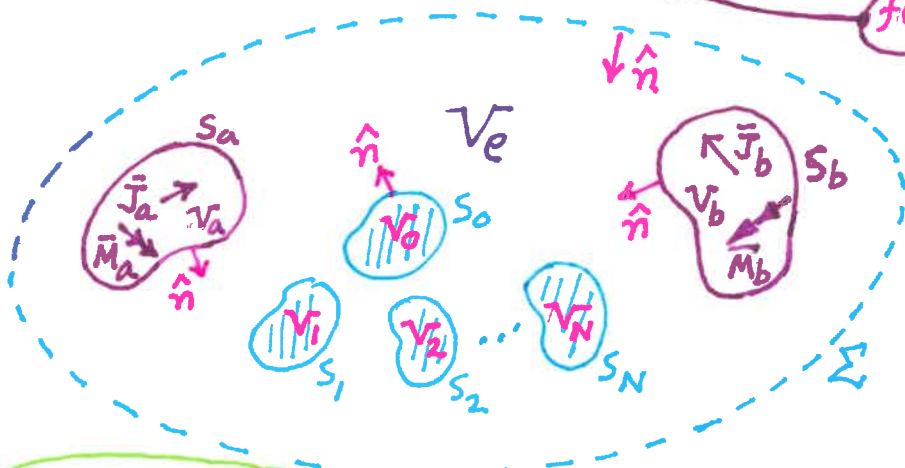
The reciprocity/reaction theorems relate the interaction of fields of problem "a" with the source of problem "b", to the interaction of the fields of problem "b" with the source of problem "a", in such a manner that a quantity called REACTION is CONSERVED.

fictitious equivalent sources

Problem "a": $\nabla\times\bar{E}_a = -\bar{M}_a - j\omega\mu\bar{H}_a \; ; \; \nabla\times\bar{H}_a = \bar{J}_a + j\omega\epsilon\bar{E}_a$.

Problem "b": $\nabla\times\bar{E}_b = -\bar{M}_b - j\omega\mu\bar{H}_b \; ; \; \nabla\times\bar{H}_b = \bar{J}_b + j\omega\epsilon\bar{E}_b$.

fictitious equivalent sources



$\hat{n}$ points into V_e

$V = V_e + V_a + V_b + \sum_{n=0}^{N} V_n$.

V is bounded only by Σ.

V_e is bounded by $\Sigma + S_a + S_b + \sum_{n=0}^{N} S_n$.

when $(\bar{J}_{\substack{a\\b}}, \bar{M}_{\substack{a\\b}})$ are turned OFF then the space within $V_{\substack{a\\b}}$ is filled with the same medium as in V_e.

The reciprocity theorem is usually developed as follows:

$$\nabla\cdot(\bar{E}_a\times\bar{H}_b - \bar{E}_b\times\bar{H}_a) = (\nabla\times\bar{E}_a)\cdot\bar{H}_b - (\nabla\times\bar{E}_b)\cdot\bar{H}_a + (\nabla\times\bar{H}_a)\cdot\bar{E}_b - (\nabla\times\bar{H}_b)\cdot\bar{E}_a$$

Let $(\bar{J}_{a,b}\,;\,\bar{M}_{a,b})$ radiate in V and with $\sum_{0}^{N} S_n$ present.

When $(\bar{J}_a, \bar{M}_a)$ are turned on, $(\bar{J}_b, \bar{M}_b)$ are turned off, and VICE VERSA.

From Maxwell's equations one obtains

$$\nabla\cdot(\bar{E}_a\times\bar{H}_b-\bar{E}_b\times\bar{H}_a)=-(\bar{E}_a\cdot\bar{J}_b-\bar{H}_a\cdot\bar{M}_b)+(\bar{E}_b\cdot\bar{J}_a-\bar{H}_b\cdot\bar{M}_a).$$

Integrating above result over V yields

$$\int_V\nabla\cdot(\bar{E}_a\times\bar{H}_b-\bar{E}_b\times\bar{H}_a)dv=-\int_{V_b}[\bar{E}_a\cdot\bar{J}_b-\bar{H}_a\cdot\bar{M}_b]dv+\int_{V_a}[\bar{E}_b\cdot\bar{J}_a-\bar{H}_b\cdot\bar{M}_a]dv$$

Div. Th

$$-\oint_\Sigma[\bar{E}_a\times\bar{H}_b-\bar{E}_b\times\bar{H}_a]\cdot\hat{n}\,ds=\underset{\sim\sim}{RHS}$$

Let $\boxed{\hat{n}\times\bar{E}_{a,b}=Z_s\,\hat{n}\times\hat{n}\times\bar{H}_{a,b}}$, be the linear impedance boundary condition on Σ which encloses V. The perfect electric boundary condition (PEC) and the perfect magnetic boundary condition (PMC) are special cases of the impedance boundary condition.

Since $(\bar{E}_a\times\bar{H}_b-\bar{E}_b\times\bar{H}_a)\cdot\hat{n}=(\hat{n}\times\bar{E}_a)\cdot\bar{H}_b-(\hat{n}\times\bar{E}_b)\cdot\bar{H}_a$

$$=[Z_s\hat{n}\times\hat{n}\times\bar{H}_a\cdot\bar{H}_b-Z_s\hat{n}\times\hat{n}\times\bar{H}_b\cdot\bar{H}_a]$$

$\therefore$

$$\bar{E}_a\times\bar{H}_b\cdot\hat{n}-\bar{E}_b\times\bar{H}_a\cdot\hat{n}=Z_s(\hat{n}\times\bar{H}_b)\cdot(\hat{n}\times\bar{H}_a)-Z_s(\hat{n}\times\bar{H}_a)\cdot(\hat{n}\times\bar{H}_b)$$

or,

$$-\oint_\Sigma[\bar{E}_a\times\bar{H}_b-\bar{E}_b\times\bar{H}_a]\cdot\hat{n}\,ds=0.$$

[This is also true if $\Sigma\to\infty$ where $\bar{H}=\hat{n}\times\bar{E}$ as $r\to\infty$ since for $\Sigma\to\infty$, $-\hat{n}\to\hat{r}$]

Thus, one obtains the Lorentz form of the **RECIPROCITY THEOREM**:

$$\int_{V_b}[\bar{E}_a\cdot\bar{J}_b-\bar{H}_a\cdot\bar{M}_b]dv=\int_{V_a}[\bar{E}_b\cdot\bar{J}_a-\bar{H}_b\cdot\bar{M}_a]dv.$$

Rumsey defined $\int_{V_2}(\bar{E}_1\cdot\bar{J}_2-\bar{H}_1\cdot\bar{M}_2)dv\equiv\langle 1,2\rangle$ where $\langle 1,2\rangle$ is the reaction of fields of source pair 1 on the source pair 2. One may therefore express the reciprocity relation above as [3]

$$\Longrightarrow\quad\langle a,b\rangle=\langle b,a\rangle.$$

(CONSERVATION OF REACTIONS)

If $\bar{M}_b = 0$, while $\bar{J}_b\, dv = d\bar{p}_{eb} = \bar{p}_{eb}\,\delta(\bar{r}-\bar{r}_b)$; likewise, if $\bar{M}_a = 0$, while $\bar{J}_a dv = d\bar{p}_{ea} = \bar{p}_{ea}\,\delta(\bar{r}-\bar{r}_a)$. Then

$$\langle a, b\rangle = \langle b, a\rangle$$

yields

$$\int_{V_b} \bar{E}_a \cdot \bar{p}_{eb}\,\delta(\bar{r}-\bar{r}_b)\,dv = \int_{V_a} \bar{E}_b \cdot \bar{p}_{ea}\,\delta(\bar{r}-\bar{r}_a)\,dv$$

or

$$\bar{E}_a(\bar{r}_b)\cdot\bar{p}_{eb} = \bar{E}_b(\bar{r}_a)\cdot\bar{p}_{ea}\,.$$

If $\bar{J}_b = 0 = \bar{M}_a$, and $\bar{J}_a = \bar{p}_{ea}\,\delta(\bar{r}-\bar{r}_b)$ while $\bar{M}_b = \bar{p}_{mb}\,\delta(\bar{r}-\bar{r}_b)$ then $\langle a, b\rangle = \langle b, a\rangle$ yields

$$\bar{E}_b(\bar{r}_a)\cdot\bar{p}_{ea} = -\bar{H}_a(\bar{r}_b)\cdot\bar{p}_{mb}\,.$$

Circuit form of the reciprocity theorem

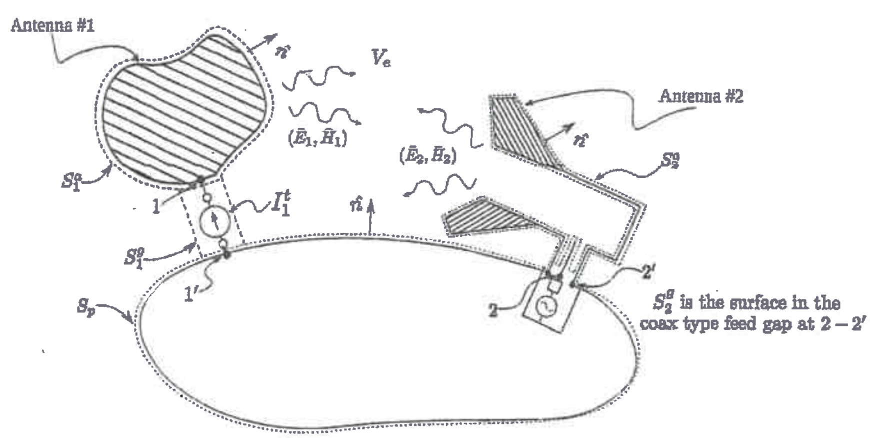


Figure A pair of antennas on a platform.

Let antenna #1 radiate in presence of antenna #2 but with the source of antenna #2 OFF. Let $(\bar{E}_1, \bar{H}_1)$ denote the fields of antenna #1 with the structure of antenna #2 present and also with the platform present.
(NOTE: The platform need not be present but is included here)

Let antenna#2 radiate $(\bar{E}_2, \bar{H}_2)$ in presence of antenna #1, but with the source of antenna#1 OFF, and in the presence of the platform.

Consider the volume V_e which is bounded by $\Sigma + S_1 + S_2 + S_p$, where S_1 and S_2 tightly encapsulates the antennas #1 and #2, respectively, and S_p tightly covers the platform. Also Σ is the surface which is allowed to receed to infinity. Next,

$$\int_{V_e} \nabla\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)\,dv = -\oint_{\Sigma+S_1+S_2+S_p}(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)\cdot\hat{n}\,ds,$$

via the divergence theorem. The integral on Σ vanishes as before. Also, it is useful to write $S_1 = S_1^a + S_1^g$, and $S_2 = S_2^a + S_2^g$. Here, $S_{1,2}^a$ are the surfaces of antenna#(1,2) and $S_{1,2}^g$ are the surfaces which bound the feed gaps of antenna#(1,2), respectively.

Since there are no sources in V_e, the L.H.S. of the above equation vanishes via Maxwell's equations. Thus,

$$\oint_{S_1+S_2+S_p}(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)\cdot\hat{n}\,ds = 0.$$

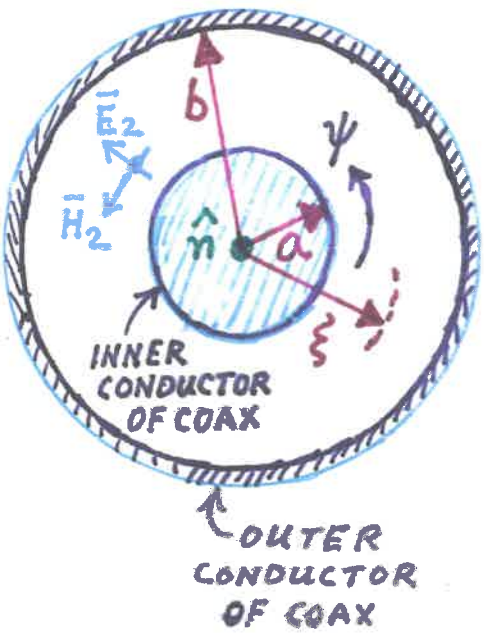
One may assume in general that $\hat{n}\times\hat{n}\times\bar{E}_{1,2} = -\bar{\bar{Z}}_s\cdot\hat{n}\times\bar{H}_{1,2}$, where $\bar{\bar{Z}}_s$ is a symmetrical dyadic surface impedance, for $[S_1^a + S_2^a + S_p]$ in which $\bar{\bar{Z}}_s$ may be different on S_1^a, S_2^a and S_p. Hence $\int_{S_1^a+S_2^a+S_p}(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)\cdot\hat{n}\,ds = 0$. Thus,

$$\int_{S_1^g+S_2^g}(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)\cdot\hat{n}\,ds \equiv 0.$$

Thus,

$$\oint_{S_1^g} (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds = \int_{S_2^g} (\bar{E}_2 \times \bar{H}_1 - \bar{E}_1 \times \bar{H}_2) \cdot \hat{n}\, ds$$

Clearly $\oint_{S_2^g} \bar{E}_2 \times \bar{H}_1 \cdot \hat{n}\, ds = \int V_2^t \bar{e} \times I_2^r \bar{h} \cdot \hat{n}\, ds$



where $\bar{e}$ and $\bar{h}$ are defined to be the DOMINANT VECTOR MODAL or TEM fields in the coax. At S_2^g it is assumed that sufficiently far from any discontinuities, only this TEM mode exists. The subscript 2 on V_2^t and I_2^r refers to the values at the terminal gap S_2^g.

$\bar{h} = \hat{n} \times \bar{e}$; $\int \bar{e} \times (\hat{n} \times \bar{e})\, ds$; $ds = \xi\, d\xi\, d\psi$.

The integral is normalized such that

$$\int_a^b \int_0^{2\pi} \bar{e} \times (\hat{n} \times \bar{e})\, ds = \int \bar{e} \cdot \bar{e}\, ds = 1.$$

Here, V_2^t is the terminal voltage at S_2^g when antenna #2 is transmitting, while I_2^r is the terminal current at S_2^g when antenna #2 receives.

The COAX GAP S_2^g is assumed to be sufficiently small so displacement currents may be neglected.

$$\therefore \quad \int_{S_2^g} \bar{E}_2 \times \bar{H}_1 \cdot \hat{n}\, ds = (V_2^t)(I_2^r).$$

Similar reasoning leads to:

$$\int_{S_2^g} \bar{E}_1 \times \bar{H}_2 \cdot \hat{n}\, ds = (V_2^r)(I_2^t).$$

Thus

$$\int_{S_2^g} (\bar{E}_2 \times \bar{H}_1 - \bar{E}_1 \times \bar{H}_2) \cdot \hat{n}\, ds = V_2^t I_2^r - V_2^r I_2^t .$$

It remains to evaluate $\oint_{S_1^g} (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds$.

Due to the different type of feed excitation in S_1^g it is convenient to convert $\int_{S_1^g} d\bar{s}\,(\cdot)$ into $\int_{V_1^g} dv(\cdot\cdot)$.

From the divergence theorem: $\oint_{S_1^g} \bar{A} \cdot \hat{n}\, ds = \int_{V_1^g} \nabla \cdot \bar{A}\, dv$

where V_1^g is the volume enclosed by S_1^g. Thus

$$\oint_{S_1^g} (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds = \int_{V_1^g} \nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1)\, dv$$

From Maxwell's equations: $\nabla \times \bar{E}_1 = -j\omega\mu \bar{H}_1$; $\nabla \times \bar{H}_1 = \bar{J}_1 + j\omega\epsilon \bar{E}_1$, and $\nabla \times \bar{E}_2 = -j\omega\mu \bar{H}_2$; $\nabla \times \bar{H}_2 = j\omega\epsilon \bar{E}_2$. Also $\bar{J}_1 \in V_1^g$.

$$\oint_{S_1^g} [\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1] \cdot \hat{n}\, ds = +\int_{V_1^g} \bar{J}_1 \cdot \bar{E}_2\, dv = +\int_0^{\ell_1} \bar{E}_2 \cdot I_1^t\, d\bar{\ell}$$

For a filament of current $\bar{J}_1\, dv \rightarrow I_1^t\, d\bar{\ell}$ $(= d\bar{p}_{e1})$.
Thus, for a current I_1^t in V_1^g when antenna #1 transmits:

$$\oint_{S_1^g} (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds = +I_1^t \int_0^{\ell_1} \bar{E}_2 \cdot d\bar{\ell} \equiv -I_1^t V_1^r .$$

where I_1^t is assumed constant over $k\ell_1 << 1$, and V_1^r is $-\int_0^{\ell_1} \bar{E}_1 \cdot d\bar{\ell}$ which is the voltage received at antenna #1 when of course antenna #2 transmits.

For a more complex or COAX FEED at antenna #1, one obtains (as for antenna #2):

$$\oint_{S_1^g} (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) \cdot \hat{n}\, ds = +V_1^t I_1^r - V_1^r I_1^t .$$

From above, it is clear that $\oint_{S_g'}(\bar{E}_1\times\bar{H}_2-\bar{E}_2\times\bar{H}_1)\cdot\hat{n}ds$

$$= \oint_{S_2^g}(\bar{E}_2\times\bar{H}_1-\bar{E}_1\times\bar{H}_2)\cdot\hat{n}ds$$

becomes:
$$-V_1^r I_1^t = V_2^t I_2^r - V_2^r I_2^t .$$

Since antenna #1 receives with the source $I_1^t=0$, it follows that V_1^r is received at antenna #1 under OPEN CKT. CONDITIONS; thus, $\boxed{V_1^r = V_1^{oc}}$. If antenna #2 also receives under open circuit conditions, $I_2^r=0;\ V_2^r=V_2^{oc}$.
∴ When antennas #1 and #2 both receive under open ckt.:

$$V_1^{oc} I_1^t = V_2^{oc} I_2^t \longrightarrow \boxed{\frac{V_1^{oc}}{I_2^t} = \frac{V_2^{oc}}{I_1^t}}$$

A transfer or MUTUAL IMPEDANCE between antennas #1 and #2 as:

$$Z_{21} \equiv \frac{V_2^{oc}}{I_1^t} \quad ; \quad Z_{12} \equiv \frac{V_1^{oc}}{I_2^t} .$$

It follows from the above that $\boxed{Z_{12} = Z_{21}}$.
The transfer impedance is SYMMETRIC or RECIPROCAL.
In general, for arbitrary feeds, the feed terminal relations are expressed as

$$V_1^t I_1^r - V_1^r I_1^t = V_2^t I_2^r - V_2^r I_2^t .$$

Under "open ckt." receiving at both antennas, $I_1^r = 0 = I_2^r$

$$V_1^r I_1^t = V_2^r I_2^t \longrightarrow V_1^{oc} I_1^t = V_2^{oc} I_2^t \rightarrow Z_{12} = Z_{21}$$

If "SHORT CKT." conditions are chosen for receiving at both antennas, then $V_1^r = 0 = V_2^r$; $I_1^r = I_1^{sc}$; $I_2^r = I_2^{sc}$.

$$V_1^t I_1^{sc} = V_2^t I_2^{sc} \longrightarrow \frac{I_1^{sc}}{V_2^t} = \frac{I_2^{sc}}{V_1^t}$$

Let MUTUAL ADMITTANCE BE DEFINED AS $Y_{12} \equiv \frac{I_1^{sc}}{V_2^t}$; $Y_{21} \equiv \frac{I_2^{sc}}{V_1^t}$

$$\boxed{Y_{12} = Y_{21}}$$

$$[V] = [Z]\,[I],$$

where

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}; \quad [Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}; \quad [I] = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

The above linear relationship between $[V]$ and $[I]$ can be explicitly written as

$$V_1 = Z_{11}\,I_1 + Z_{12}\,I_2,$$

$$V_2 = Z_{21}\,I_1 + Z_{22}\,I_2.$$

NOTE: it is clear that there are only three independent parameters, Z_{11}, $Z_{12} = Z_{21}$, and Z_{22}, respectively. A T-section of impedance elements corresponding to above eqns. follows directly as shown in Figures . It follows from above eqns that the elements Z_{ij} are open-circuit

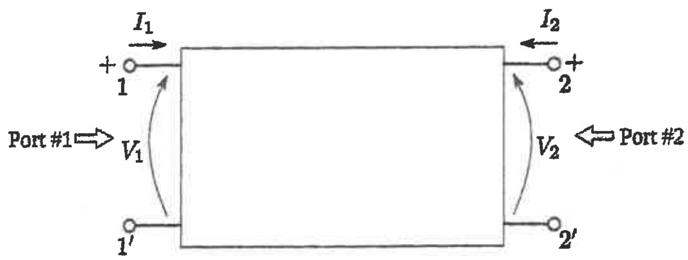


Figure A two port network describing the coupling between voltages and currents at terminals 1 – 1′ and 2 – 2′.

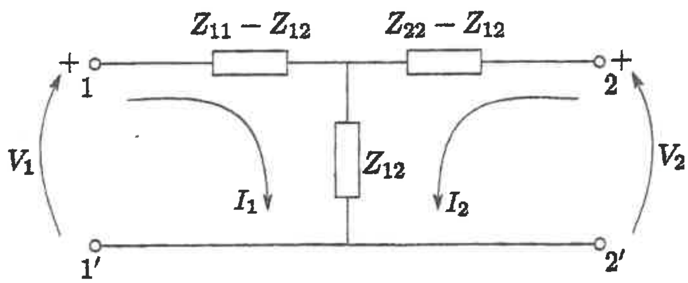


Figure Impedance elements forming a T-section representation.

Also:

$$[I] = [Y][V] \quad ; \quad [Y] = [Z]^{-1}$$

$$[Y] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

It is clear that elements of [Z] are "open ckt." parameters; likewise elements of [Y] are "short ckt." parameters.

A generalized reciprocity theorem for calculating mutual impedance between antennas [1]

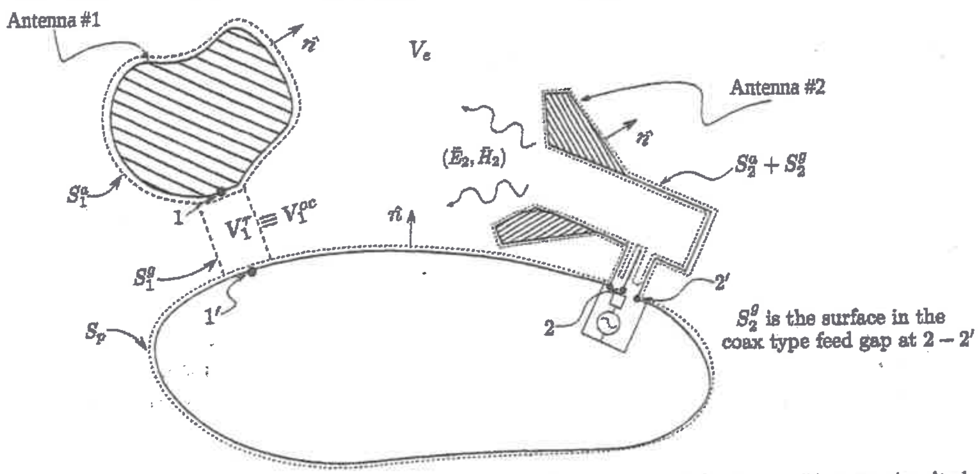


Figure A pair of antennas on a platform. Antenna #2 transmits with antenna #1 open-circuited. V_e is bounded by S_1 $(= S_1^a + S_1^g)$, S_2 $(= S_2^a + S_2^g)$, S_p, and Σ (at infinity).

ORIGINAL PROBLEM

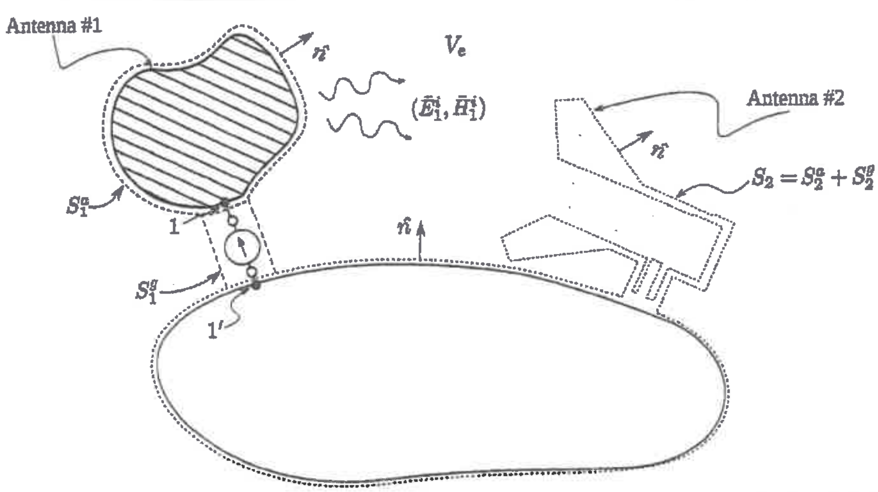


Figure Antenna #1 transmits on a hypothetical smooth platform. It is noted that antenna #2 is removed. An ideal current source I_1^i is used across $1-1'$ to excite antenna #1.

TEST PROBLEM
(generalized reciprocal problem)

Antenna #2 transmits $(\bar{E}_2, \bar{H}_2)$ in presence of antenna #1 which is OPEN CKTD for receiving.
Next let antenna #1 transmit $(\bar{E}_1^i, \bar{H}_1^i)$ with antenna #2 ABSENT.

Both antennas radiate in the presence of the platform S_p.
The original and test problems are defined within the same volume V_e bounded by

$$S + \Sigma = S_1 + S_2 + S_p + \Sigma .$$

Although V_e is the same for the original and test problems, the configurations within V_e are different for the two situations (- hence the relation between the two cases is based on <u>GENERALIZED</u> RECIPROCITY).
From divergence theorem:

$$\int_{V_e} \nabla \cdot (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) dv = - \oint_{S+\Sigma} (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) \cdot \hat{n}\, ds .$$

Since there are NO sources in V_e, one can show via Maxwell's equations for $(\bar{E}_1^i, \bar{H}_1^i)$ and $(\bar{E}_2, \bar{H}_2)$ that

$$\int_{V_e} \nabla \cdot (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) dv = 0 .$$

Also the integral on $\Sigma \to \infty$ vanishes as before. Likewise, if $S_1 = S_1^a + S_1^g$ and $S_2 = S_2^a + S_2^g$ as before, it is clear that $\hat{n} \times \hat{n} \times \bar{E} = -\bar{\bar{Z}}_s \cdot \hat{n} \times \bar{H}$ on S_1^a, S_2^a, and S_p, and so integral on S_p (platform) vanishes. Note that the symmetric dyadic $\bar{\bar{Z}}_s$ may take on different values on S_1^a, S_2^a, and S_p, respectively. Thus, from above it is seen that $(\bar{E}_2, \bar{H}_2)$ satisfies boundary conditions on $S_1^a + S_2^a + S_p$, but $(\bar{E}_1^i, \bar{H}_1^i)$ satisfies boundary conditions only on $S_1^a + S_p$, but not on S_2^a.

$$\therefore \oint_{S_1^a + S_1^g + S_2 + \Sigma} (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) \cdot \hat{n}\, ds = 0$$

S_1

Since $(\bar{E}_2, \bar{H}_2)$ know about S_1^a, S_2^a, S_p and Σ while $(\bar{E}_1^i, \bar{H}_1^i)$ knows only about S_1^a, S_p and Σ it is clear that the preceeding integral yields

$$\int_{S_1^g} (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) \cdot \hat{n}\, ds = \oint_{S_2} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2) \cdot \hat{n}\, ds ,$$

$\Downarrow$ Div. Th.

$$\int_{V_1^g} \nabla \cdot (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i)\, dv = \oint_{S_2} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2) \cdot \hat{n}\, ds$$

$\Downarrow$ Maxwell's eqns.

$$\int_{V_1^g} \bar{J}_1 \cdot \bar{E}_2\, dv = I_1^t \int_0^{l_1} \bar{E}_2 \cdot d\bar{l} = \oint_{S_2} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2) \cdot \hat{n}\, ds$$

AS BEFORE

$$-V_1^r I_1^t = \oint_{S_2} [\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2] \cdot \hat{n}\, ds = \langle 2,1 \rangle = \langle 1,2 \rangle$$

Since antenna #1 receives $(\bar{E}_2, \bar{H}_2)$ with the source of antenna #1 turned off. (OPEN CKT. AT V_1^g), $\boxed{V_1^r = V_1^{oc}}$.

$$\therefore \quad V_1^{oc} = \frac{-1}{I_1^t} \oint_{S_2} [\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2] \cdot \hat{n}\, ds = -\frac{\langle 2,1 \rangle}{I_1^t}$$

$$Z_{12} = \frac{V_1^{oc}}{I_2^t} = \frac{-1}{I_1^t I_2^t} \oint_{S_2 = S_2^a + S_2^g} [\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2] \cdot \hat{n}\, ds = \frac{-\langle 2,1 \rangle}{I_1^t I_2^t}$$

$\Rightarrow$ If S_2^a is PEC $\left(\bar{\bar{Z}}_s = 0 \rightarrow \hat{n} \times \bar{E}_2 |_{S_2^a} = 0\right)$, then

$$\rightarrow Z_{12} \cong \frac{-1}{I_1^t I_2^t} \int_{S_2^a} (\bar{E}_1^i \cdot \bar{J}_{S2})\, ds ,$$

(if the contribution from the tiny gap S_2^g is negligible in comparison to S_2^a).

$\bar{J}_{S2} \equiv \hat{n} \times \bar{H}_2$

Example: Mutual Impedance between TWO Parallel Half Wavelength Dipoles

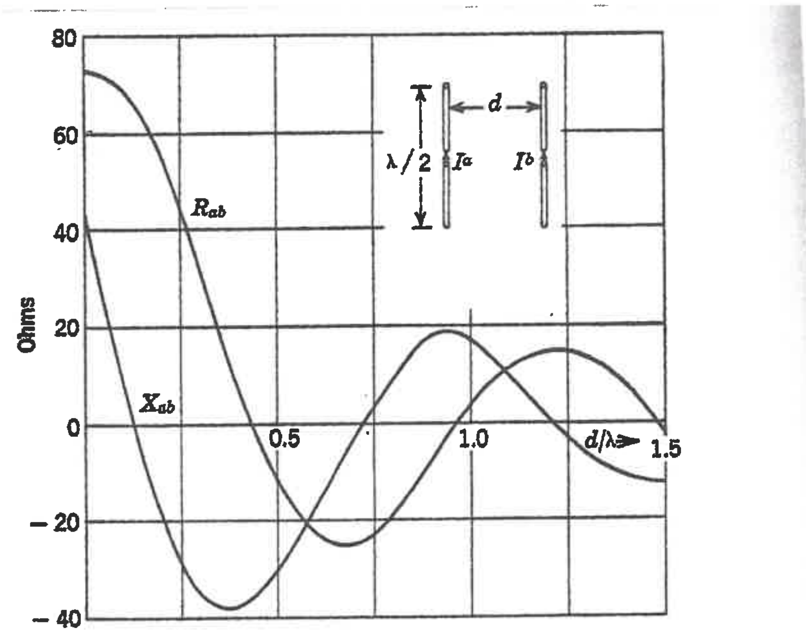


FIG. Mutual impedance $Z_{ab} = R_{ab} + jX_{ab}$ between parallel $\lambda/2$ linear antennas in free space. [3]

Let the current in one dipole be I^a, and the current in the other dipole be I^b. For arbitrary long wires, one must resort to an integral equation based solution for the currents on the coupled set of wires. However for a "thin wire" center fed dipole it is possible to "assume" a current $I(z)$ given by, $I(z) \cong \frac{I_0}{\sin\frac{kL}{2}} \sin k\left(\frac{L}{2} - |z|\right)$.

For the special case of a half wavelength dipole, $L = \frac{\lambda}{2}$ where λ = wavelength. Thus $I(z) \simeq \frac{I_0}{1} \sin\left[\frac{\pi}{2} - k|z|\right]$.

Let : $I_a \equiv I_{0a} \sin\left[\frac{\pi}{2} - k|z|\right]$; $I_b \equiv I_{0b} \sin\left[\frac{\pi}{2} - k|z|\right]$.

Next, Z_{ab} which is the mutual impedance between dipoles "a" and "b" is given by

$$Z_{ab} \cong \frac{-1}{I_{0a} I_{0b}} \cdot \int_{-L/2}^{L/2} E_a^i \cdot \hat{z}\, I_{0b} \sin\left(\frac{\pi}{2} - k|z_b|\right) dz_b$$

The $\bar{E}_a^i$ (on wire "b") is the electric field intensity at dipole "b" which is produced by I_a on antenn "a". It is easily verified from earlier notes that

$$\hat{z} \cdot \bar{E}_a^i = j\omega\mu \int_{-L/2}^{L/2} G_{ozz} \, I_{oa} \sin\left(\frac{\pi}{2} - k|z_a|\right) dz_a \; .$$

where

$$j\omega\mu \, G_{ozz} = \frac{jkZ}{4\pi}\left(-\left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial z_b^2}\right]\right)\frac{e^{-jkR}}{R}$$

$$\therefore \; Z_{ab} \approx \frac{-1}{I_{oa} I_{ob}} \cdot \int_{-L/2}^{L/2} dz_b \int_{-L/2}^{L/2} dz_a \, I_b(z_b)\left[\frac{1}{j\omega\epsilon}\left(k^2 + \frac{\partial^2}{\partial z_b^2}\right)\frac{e^{-jkR}}{R}\right] I_a(z_a).$$

(NOTE: $\frac{j\omega\mu}{k^2} = -\frac{1}{j\omega\epsilon}$; $k \equiv \omega\sqrt{\mu\epsilon}$)

Here,

$$\frac{e^{-jkR}}{R} = \frac{e^{-jk\sqrt{d^2 + (z_b - z_a)^2}}}{\sqrt{d^2 + (z_b - z_a)^2}}$$

The above can be evaluated in terms of Si and Ci functions as done previously by Carter, Richmond, and Elliot, etc.

A generalized reciprocity theorem for calculating mutual admittance between antennas [1]

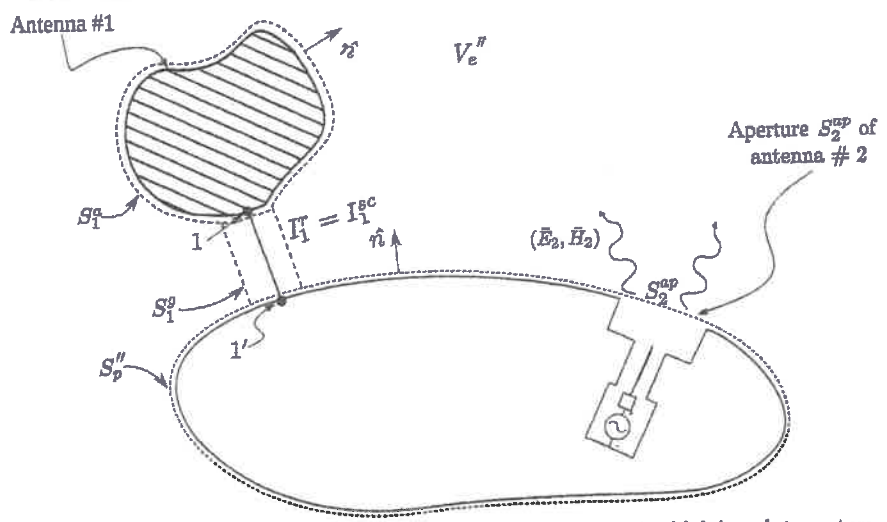


Figure A pair of antennas on a hypothetical platform. Antenna #2 which is a slot aperture, radiates with antenna #1 short-circuited.

ORIG. PROBLEM : Let antenna #2 radiate $(\bar{E}_2, \bar{H}_2)$ when it acts as a transmitter, while antenna #1 receives under SHORT CKT conditions. Note: $(\bar{E}_2, \bar{H}_2)$ radiates with the platform present and antenna #1 SHORT CKTD. The platform surface is S_p''. The antenna #2 slot aperture is S_2^{ap}.

TEST PROBLEM : Let antenna #1 radiate $(\bar{E}_1^i, \bar{H}_1^i)$ when it acts as a transmitter with the aperture S_2^{ap} NOW CLOSED so antenna #1 is essentially "removed".

Let the external volume V_e'' be bounded by the closed surface $S_1 + S_2^{ap} + S_p'' + \Sigma$ (with $\Sigma \to \infty$). Also, $S_1 = S_1^a + S_1^g$ as before. Thus, one may write

$$\int_{V_e''} \nabla \cdot (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i)\,dv = -\oint_{S_1 + S_2^{ap} + S_p'' + \Sigma} (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i) \cdot \hat{n}\,ds.$$

Since there are no sources in V_e'', Maxwell's equations make the integral on V_e'' on the LHS vanish. Thus,

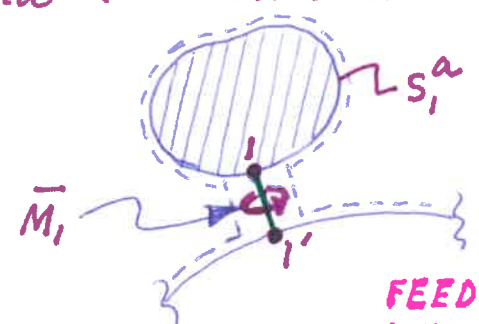
$$\oint_{S_1^g} (\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i)\cdot\hat{n}\,ds = \int_{S_2^{ap}} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2)\cdot\hat{n}\,ds$$

since $(\bar{E}_1^i, \bar{H}_1^i)$ and $(\bar{E}_2, \bar{H}_2)$ satisfy the same boundary conditions on S_1^a and S_p'', while the contribution on $\Sigma \to \infty$ vanishes as usual.

From the divergence theorem applied to the integral over S_1^g on the LHS of the preceeding equation one obtains

$$\int_{V_1^g} \nabla\cdot(\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i)\,dv = \int_{S_2^{ap}} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2)\cdot\hat{n}\,ds$$

Maxwells equations in V_1^g are $\nabla\times\bar{E}_1^i = -\bar{M}_1 - j\omega\mu\bar{H}_1^i$, $\nabla\times\bar{H}_1^i = j\omega\epsilon\bar{E}_1^i$; $\nabla\times\bar{E}_2 = -j\omega\mu\bar{H}_2$, and $\nabla\times\bar{H}_2 = j\omega\epsilon\bar{E}_2$. Here, $\bar{M}_1 \equiv -V_1^t\hat{c}$, which is a tiny loop of magnetic current which is an equivalent impressed source for antenna #1 when antenna #1 transmits.



FEED SOURCE $\bar{M}_1$ WHEN ANT #1 ACTS AS XMTR.

If $\bar{M}_1 = 0$ then only the short ckt. wire remains between terminals 1-1'.

Thus, $\int_{V_1^g} \nabla\cdot(\bar{E}_1^i \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1^i)\,dv = -\int_{V_1^g} \bar{M}_1\cdot\bar{H}_2\,dv = +V_1^t \oint \bar{H}_2\cdot d\bar{c}$.

Since $\bar{M}_1 = -V_1^t\hat{c}$ and $\oint \bar{H}_2\cdot d\bar{c} = I_1^r$ (via Amperes Th.) if the displacement currents are neglected within the tiny loop $d\bar{c}$. Thus, the expression at the very top yields:

$$V_1^t I_1^r \equiv V_1^t I_1^{sc} = \int_{S_2^{ap}} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2)\cdot\hat{n}\,ds$$

since $I_1^r = I_1^{sc}$ with $\bar{M}_1 = 0$ when antenna #1 is RCVR.

When antenna #1 acts as a XMTR, $\bar{M}_1$ acts as an ideal voltage generator and drives a current I_1^t through terminals 1 - 1′.
When antenna #1 acts as a RCVR, $\bar{M}_1$ is turned OFF, i.e., $\bar{M}_1 \equiv 0$ and the wire connecting 1-1′ is only left there constituting a SHORT CKT., so $I_1^r \equiv I_1^{sc}$.

Thus,

$$I_1^{sc} = \frac{1}{V_1^t} \int_{S_2^{ap}} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2) \cdot \hat{n}\, ds.$$

and

$$Y_{12} = \frac{I_1^{sc}}{V_2^t} = \frac{1}{V_1^t V_2^t} \int_{\substack{S_2^{ap} \\ (SLOT)}} (\bar{E}_2 \times \bar{H}_1^i - \bar{E}_1^i \times \bar{H}_2) \cdot \hat{n}\, ds.$$

A useful special situation occurs where S_p'' is a PEC surface, so that antenna #1 transmits with S_2^a NOW CLOSED BY A PEC on which $\hat{n} \times \bar{E}_1^i \big|_{S_2^a} = 0$.
In this situation, the external aperture-aperture coupling:

$$Y_{12} = \frac{1}{V_1^t V_2^t} \int_{S_2^{ap}} \bar{E}_2 \times \bar{H}_1^i \cdot \hat{n}\, ds$$

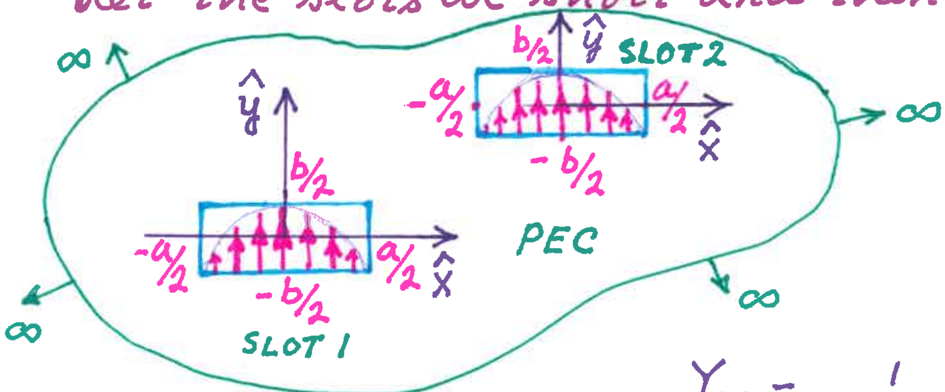
OR

$$Y_{12} = \frac{-1}{V_1^t V_2^t} \int_{S_2^{ap}} \bar{M}_{S2} \cdot \bar{H}_1^i\, ds \; ; \; \bar{M}_{S2} \equiv \bar{E}_2 \times \hat{n}$$

The above expression remains valid even if antenna #1 is a slot type antenna.

Example: Mutual Admittance between TWO Parallel Rectangular Slots in a Planar PEC Surface.

Let the slots be "short" and "thin" as shown below:



One can assume:

$$\bar{E}_{a1} = \hat{y}\, V_1 \left(\sqrt{\frac{2}{ab}} \cos\frac{\pi x_1}{a}\right)$$

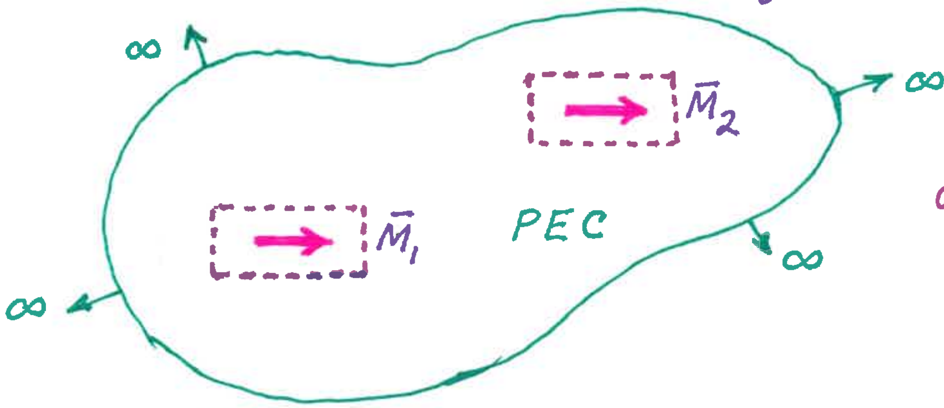
$$\bar{E}_{a2} = \hat{y}\, V_2 \left(\sqrt{\frac{2}{ab}} \cos\frac{\pi x_2}{a}\right)$$

$$Y_{12} = -\frac{1}{V_1 V_2} \int_{S_2^{ap}} \bar{H}_1^i \cdot \bar{M}_2 \, ds_2$$

where, $a > b$, and

$$\bar{M}_2 \equiv \bar{E}_{a2} \times \hat{z} = \hat{x} M_2$$

$\bar{H}_1^i$ is the magnetic field produced by slot 1, which arrives within the region of slot 2, but with slot 2 shorted (i.e. covered by PEC). From an "equivalence theorem", the slots can be replaced by equivalent MAGNETIC SURFACE CURRENTS on the PEC ground plane.



EQUIVALENT PROBLEM

$$\bar{M}_1 = \hat{x}\, V_1 \sqrt{\frac{2}{ab}} \cos\frac{\pi}{a} x_1$$

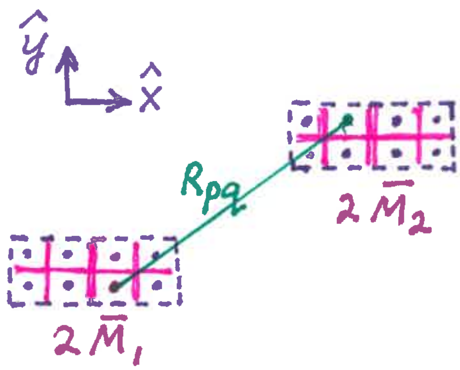
$$\bar{M}_2 = \hat{x}\, V_2 \sqrt{\frac{2}{ab}} \cos\frac{\pi}{a} x_2$$

with

$$\bar{M}_1 \equiv \bar{E}_{a1} \times \hat{z}$$

$$\bar{M}_2 \equiv \bar{E}_{a2} \times \hat{z}$$

A further simplification occurs via IMAGE THEORY, namely the PEC ground plane may be removed and the "impressed equivalent current sources" then must be doubled in strength.



REDUCTION BY USE OF IMAGE THEORY

$$Y_{12} = \frac{-1}{V_1 V_2} \int_{S_2^a} \bar{H}_1 \cdot 2\bar{M}_2 \, ds_2$$

$\bar{H}_1 = 2\bar{H}_1^i$ via image theory

From duality: $\bar{H}_1 = j\omega\epsilon \int_{S_1^a} \bar{\bar{G}}_0 \cdot 2\bar{M}_1 \, ds_1$.

Also, $\bar{H}_1$ above is evaluated on S_2^a with $\bar{M}_2 = 0$.
Thus, the external aperture-aperture coupling is:

$$Y_{12} = \frac{-4}{V_1 V_2} \int_{S_2^a} ds_2 \int_{S_1^a} ds_1 \, \bar{M}_2 \cdot \bar{\bar{G}}_0 \cdot \bar{M}_1$$

or

$$Y_{12} = -4 \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} dx_2\, dy_2 \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} dx_1\, dy_1 \left(\frac{2}{ab}\right) \cos\frac{\pi x_2}{a} \left[\hat{x}\cdot\bar{\bar{G}}_0\cdot\hat{x}\right] \cos\frac{\pi x_1}{a}$$

$\therefore$

$$Y_{12} \approx \frac{-8}{ab} \sum_{q=1}^{Q} \Delta_q \sum_{p=1}^{P} \Delta_p \cdot \cos\frac{\pi x_q}{a} \, G_{oxx} \cos\frac{\pi x_p}{a} .$$

$$G_{oxx}(q;p) = -\left[1 + \frac{1}{k^2}\frac{\partial^2}{\partial x_p^2}\right] \frac{e^{-jkR_{pq}}}{4\pi R_{pq}}$$

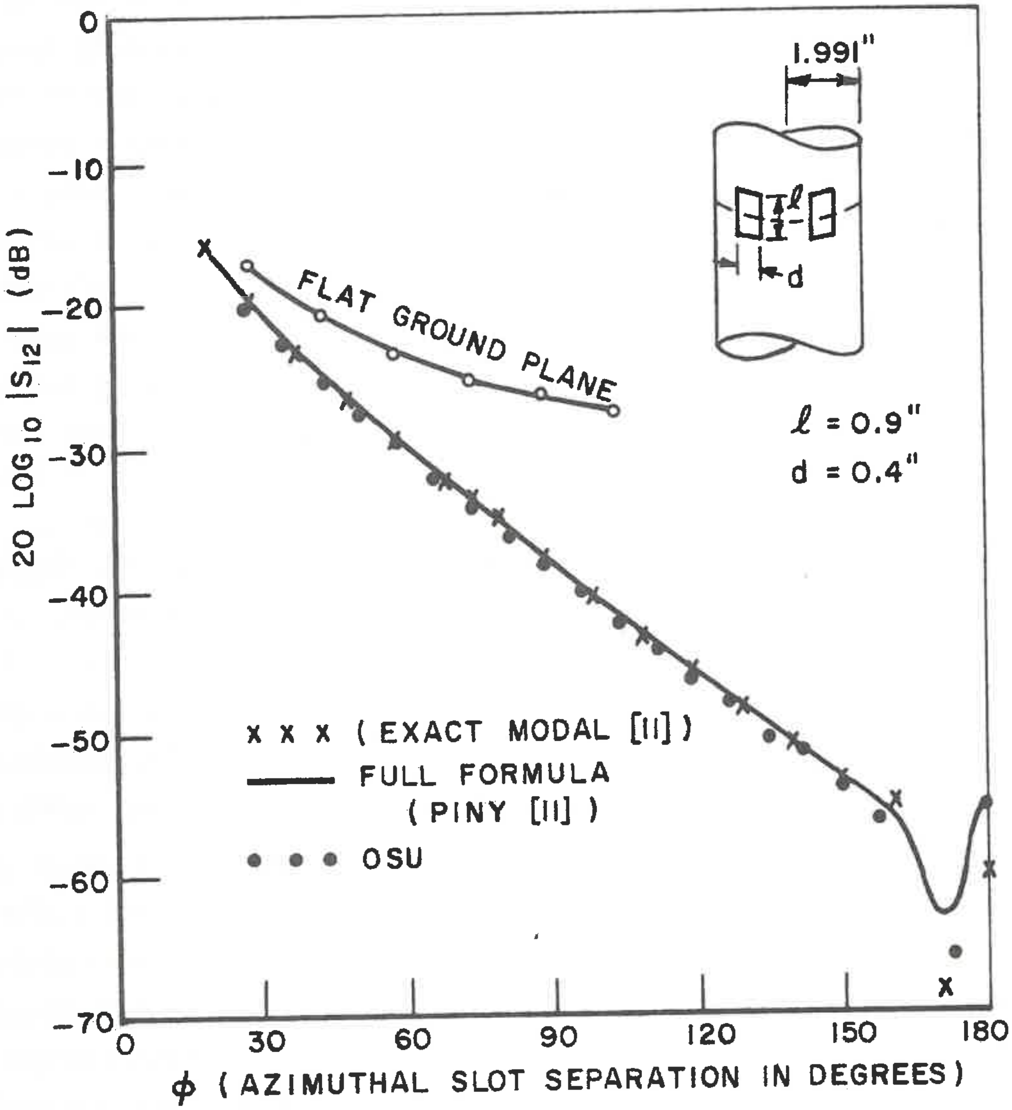


Figure Isolation of axial slots on a conducting cylinder, a = 1.991"; Z_0 = 0; Frequency = 9 GHz.

[11] Felsen et. al.

OSU → Pathak and Wang.

The receiving antenna problem

One can introduce a Thevenin equivalent CKT. for the receiving antenna problem as shown below, together with its alternative Norton equivalent CKT.

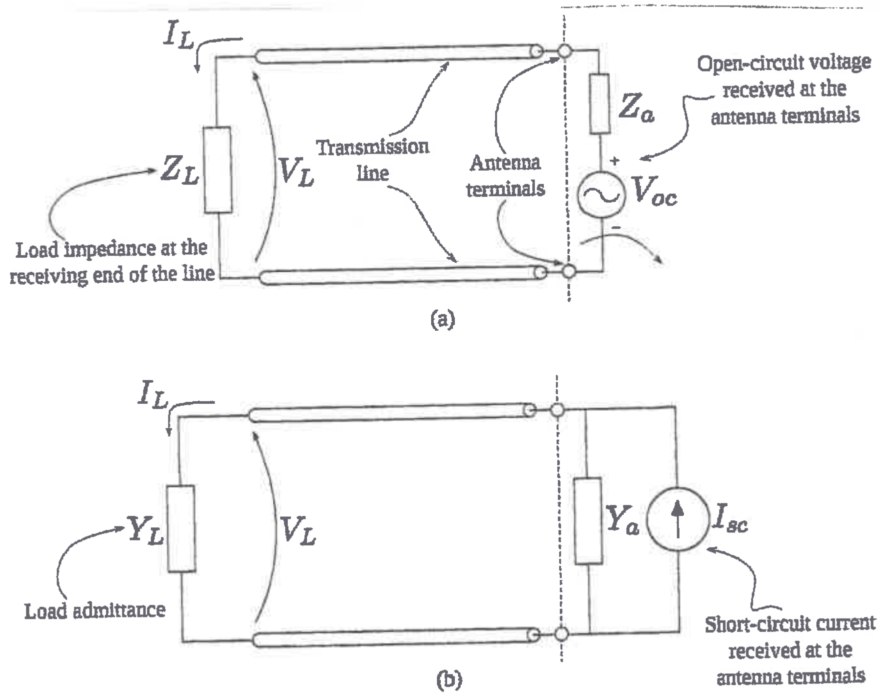


Figure An equivalent circuit for an antenna under receiving conditions. The received open-circuit voltage, or the received short-circuit current, respectively, at the antenna terminals produces a load voltage V_L and a load current I_L at the load terminals. (a) denotes a Thevenin equivalent circuit for the antenna, while (b) denotes a Norton equivalent circuit for the antenna, respectively.

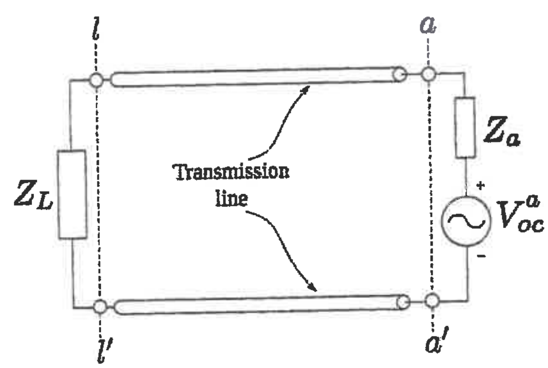


Figure Equivalent circuit for a receiving antenna. The receiving antenna terminals are at $a - a'$. This circuit is the same as Figure 6.41(a), but shows the antenna and load terminal locations as $a - a'$ and $\ell - \ell'$, respectively. The open-circuit voltage V_{oc}^a received by the antenna terminals is of interest.

The antenna impedance concept has been introduced earlier. It is therefore of interest to find V_{oc} (OPEN CKT. RECVD. VOLTAGE) at the RCVR antenna terminals for the Thevenin CKT., or find I_{sc} (SHORT CKT. RECVD. CURRENT) for the Norton CKT. NOTE: $V_{oc} = I_{sc} Z_a$ or $I_{sc} = V_{oc} Y_a$ and $Y_a = Z_a^{-1}$. Specifically,

V_{oc} = OPEN CKT. voltage at the antenna terminals a-a′ which is received after it is illuminated by another "distant antenna". It is assumed that the antenna impedance Z_a is known. Also the load Z_L is known. Let

$$V_{oc}(\text{at } a\text{-}a') \equiv V_{oc}^{a} \quad .$$

In particular let the receiving antenna "a" be excited by a "distant" transmitting antenna "b". This situation is referred to as the ORIGINAL PROBLEM where it is required to find V_{oc}^{a} .

The above V_{oc}^{a} can be found in terms of the FAR ZONE radiation pattern of the antenna "a" itself when it acts as a XMTR. This fact is not surprising since it is known that the radiation and receiving patterns of reciprocal antennas are directly related as can be proved via the reciprocity theorem $(Z_{12} = Z_{21})$. No specific knowledge of the type of antenna "b" is necessary to find V_{oc}^{a}; only the strength of the field incident on

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Next one introduces a TEST (or generalized reciprocal) PROBLEM in which antenna "a" radiates in the absence of antenna "b" (i.e. antenna "b" is removed).

The ORIGINAL and TEST problems are shown below[1].

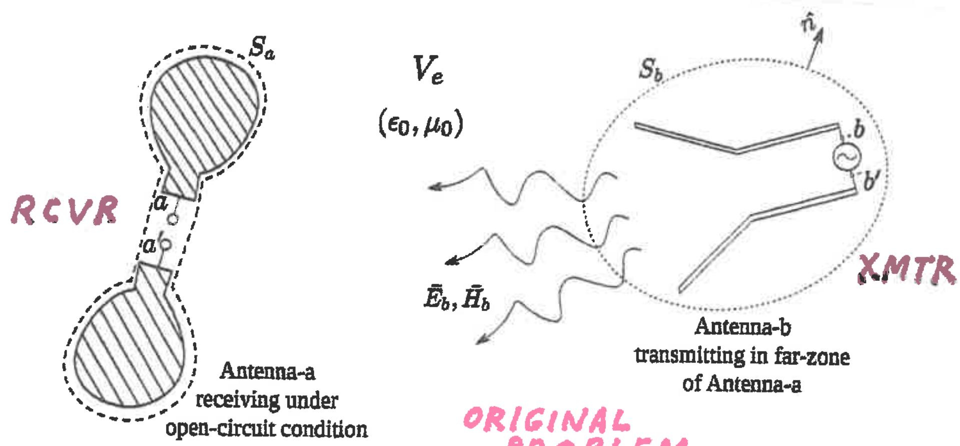


Figure Antenna-a receives under open-circuit conditions when Antenna-b transmits. Antenna-b is in the far zone of Antenna-a, and vice versa. The closed surfaces S_a and S_b enclose Antennas-a and -b, respectively. The medium external to the antennas is free space. The volume V_e is bounded by the surfaces $S_a + S_b + \Sigma$ where Σ is a spherical surface at unity.

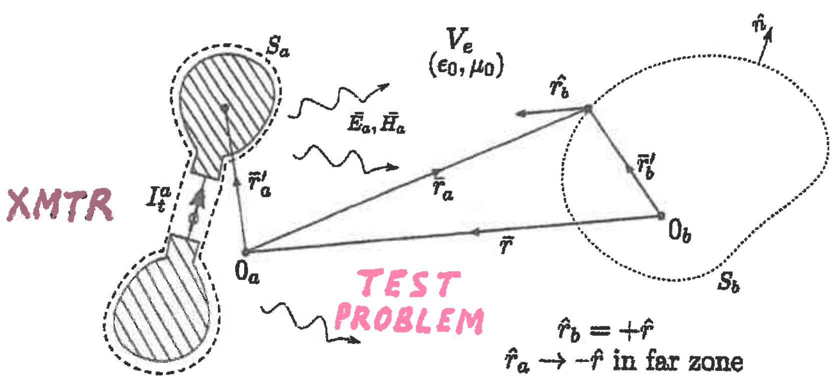


Figure A generalized reciprocal problem such that Antenna-a transmits in the absence of Antenna-b.

Let antenna "b" transmit $(\bar{E}_b, \bar{H}_b)$ with antenna "a" receiving under OPEN CKT CONDITIONS in the ORIGINAL PROBLEM.

Let antenna "a" transmit $(\bar{E}_a, \bar{H}_a)$ with ant "b" absent in the TEST PROBLEM.

Let the external volume V_e be bounded by $S_a + S_b + \Sigma$ (with $\Sigma \to \infty$). Here S_a tightly bounds antenna "a", so that $S_a = S_c^a + S_g^a$, where S_c^a is the antenna "a" structure and S_g^a is the feed gap of antenna "a". Also, S_b is a conveniently chosen surface (like a bubble) enclosing antenna "b".

$$\int_{V_e} \nabla \cdot (\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a) \cdot \hat{n}\, ds = 0$$

since there are no sources in V_e. From the divergence th.:

$$-\oint_{S_a + S_b + \Sigma} (\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a) \cdot \hat{n}\, ds = 0$$

Antenna "a" knows about $S_c^a + \Sigma$ in the TEST PROBLEM, while antenna "b" knows about $S_c^a + \Sigma$ in the ORIGINAL PROBLEM. Note that antenna "a" has no knowledge of antenna "b" in the TEST PROBLEM. Also, $\Sigma \to \infty$. As before there is no contribution to the above integral from Σ. Additionally, there is no contribution from S_c^a either (since both antennas know about S_c^a). THUS only the contributions from S_g^a and S_b remain:

$$\oint_{S_g^a} (\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a) \cdot \hat{n}\, ds = \oint_{S_b} (\bar{J}_{sb}^{eq} \cdot \bar{E}_a - \bar{M}_{sb}^{eq} \cdot \bar{H}_a)\, ds$$

$$\boxed{\bar{J}_{sb}^{eq} \equiv \hat{n} \times \bar{H}_b \; ; \quad \bar{M}_{sb}^{eq} \equiv \bar{E}_b \times \hat{n}}$$

Clearly $(\bar{J}_{sb}^{eq}, \bar{M}_{sb}^{eq})$ are the "equivalent sources" of antenna "b" which reside on S_b^+ and produce $(\bar{E}_b, \bar{H}_b)$ with antenna "b" now removed.

As before the integral on S_g^a can be written via Div. Th. as:

$$\oint_{S_g^a} (\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a)\cdot\hat{n}\, ds = \int_{V_g^a} \nabla\cdot(\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a)\, dv$$

and from Maxwell's equations $\int_{V_g^a} \nabla\cdot(\bar{E}_a \times \bar{H}_b - \bar{E}_b \times \bar{H}_a)\, dv = I_t^a \int_{a'}^{a} \bar{E}_b \cdot d\bar{l}$.

One thus obtains:

$$-I_t^a V_{oc}^a = \oint_{S_b} [\bar{J}_{sb}^{eq} \cdot \bar{E}_a - \bar{M}_{sb}^{eq} \cdot \bar{H}_a]\, ds = \langle a, b \rangle$$

→ or $$\boxed{V_{oc}^a = -\frac{\langle a, b\rangle}{I_t^a}},$$

where I_t^a is the transmit mode current when antenna "a" acts as an XMTR with ant. "b" removed, as in the test problem.

A simplification to the quantity $\langle a, b \rangle$ above can be obtained if antenna "b" is in the FAR ZONE of antenna "a", and VICE VERSA.

At $\bar{r}_a$, in the far zone of ant. "a"; the field radiated by ant. "a", with ant. "b" ABSENT, one can write [1]

→ $$\bar{E}_a(\bar{r}_a) \sim j\frac{kZ}{4\pi} I_t^a \bar{h}_a(\hat{r}_a) \frac{e^{-jkr_a}}{r_a}$$

→ $$\bar{h}_a(\hat{r}_a) \equiv \frac{1}{I_t^a}\left[\hat{r}_a \times \hat{r}_a \times \int_{S_a} d\bar{p}_e(\bar{r}_a')\, e^{jk\hat{r}_a\cdot\bar{r}_a'} + Y \hat{r}_a \times \int_{S_a} d\bar{p}_m(\bar{r}_a')\, e^{jk\hat{r}_a\cdot\bar{r}_a'}\right]$$

in which $d\bar{p}_e(\bar{r}_a') = \bar{J}_{sa}^{eq}(\bar{r}_a')\, ds'$ and $d\bar{p}_m(\bar{r}_a') = \bar{M}_{sa}^{eq}(\bar{r}_a')\, ds'$ represent the "equivalent sources" of ant. "a" which radiate $(\bar{E}_a, \bar{H}_a)$ when placed on S_a^+ with ant. "a" removed.

→ $$\bar{H}_a(\bar{r}_a) \sim \frac{1}{Z} \hat{r}_a \times \bar{E}_a = \frac{jk}{4\pi} I_t^a \hat{r}_a \times \bar{h}_a(\hat{r}_a) \frac{e^{-jkr_a}}{r_a}.$$

(NOTE: $\bar{h}_a$ is often referred to as ANTENNA HEIGHT).

$$\langle a,b\rangle = \frac{jkZ}{4\pi} I_t^a \oint_{S_b} \left[\bar{J}_{sb}^{eq} \cdot \bar{h}_a(\hat{r}_a) - \bar{M}_{sb}^{eq} \cdot Y \hat{r}_a \times \bar{h}_a(\hat{r}_a) \right] \frac{e^{-jkr_a}}{r_a}$$

NOTE :

$$\bar{r}_a = -\bar{r} + \bar{r}_b' \; ; \; r_a = \sqrt{\bar{r}_a \cdot \bar{r}_a} = \sqrt{(r_b')^2 + r^2 - 2\bar{r}_b' \cdot \bar{r}}$$

$\therefore$

$$r_a \approx \begin{bmatrix} r - \hat{r} \cdot \bar{r}_b' & , \text{ IN PHASE TERMS} \\ r & , \text{ IN AMPLITUDE TERMS} \end{bmatrix}$$

Incorporating the above FAR ZONE (parallel ray) APPROX. into $\langle a,b\rangle$, and simplifying the above integrand via

$$\bar{J}_{sb}^{eq} \cdot \bar{h}_a(\hat{r}_a) = -(\hat{r}_a \times \hat{r}_a \times \bar{h}_a(\hat{r}_a)) \cdot \bar{J}_{sb}^{eq} = \bar{h}_a(\hat{r}_a) \cdot \hat{r}_a \times \hat{r}_a \times \bar{J}_{sb}^{eq}$$

and

$$-\bar{M}_{sb}^{eq} \cdot \hat{r}_a \times \bar{h}_a(\hat{r}_a) = \bar{h}_a(\hat{r}_a) \cdot \hat{r}_a \times \bar{M}_{sb}^{eq}$$

as well as

$$\hat{r}_a \rightarrow -\hat{r} \quad \text{in FAR ZONE}$$

yields

$$\rightarrow V_{oc}^a = -\frac{\langle a,b\rangle}{I_t^a} = -\bar{h}_a(-\hat{r}) \cdot \left[\frac{jkZ}{4\pi} \left(\oint_{S_b} \left[\hat{r} \times \hat{r} \times \bar{J}_{sb}^{eq} + Y \hat{r} \times \bar{M}_{sb}^{eq} \right] \right) e^{-jk\hat{r} \cdot \bar{r}_b'} ds \right]$$

$\bar{E}_b(O_a)$

$\therefore$

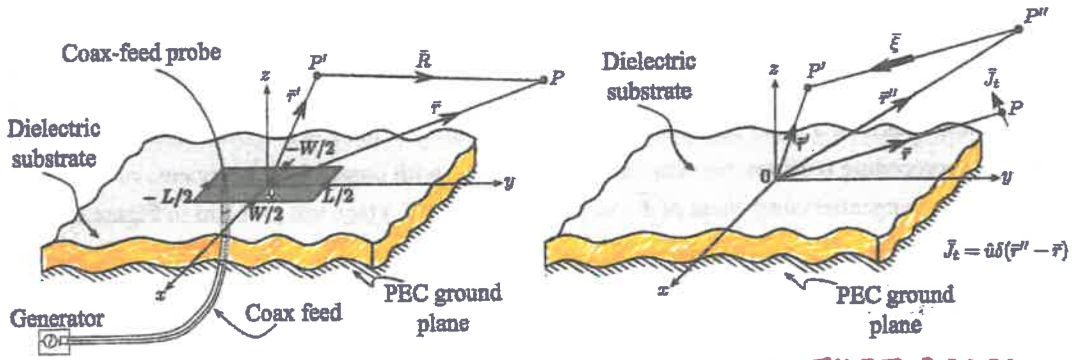
$$V_{oc}^a = -\bar{h}_a(-\hat{r}) \cdot \bar{E}_b(O_a) .$$

Since the field $\bar{E}_b = \bar{E}_b^i + \bar{E}_b^s$ (with $\bar{E}_b^i$ incident from ant. "b" onto ant. "a", and $\bar{E}_b^s$ is scattered by ant. "a"), one may assume that $(\bar{J}_{sb}^{eq}, \bar{M}_{sb}^{eq})$ are NOT MUCH PERTURBED from their values if ant. "a" was removed. Thus,

$$\rightarrow \boxed{V_{oc}^a \approx -\bar{h}_a(-\hat{r}) \cdot \bar{E}_b^i(O_a)}$$

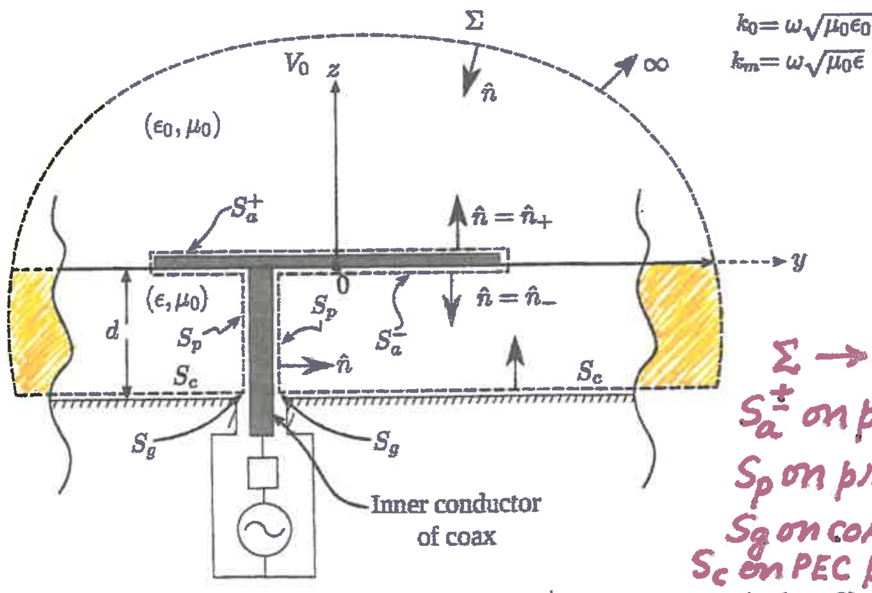
NOTE : O_a should be kept close to or within antenna "a". The above V_{oc}^a can be used in the RCVR Thevenin CKT.

An analysis of the radiation by a patch antenna without the use of microstrip Green's function [1]



(a) ORIGINAL PROBLEM (b) TEST PROBLEM

Figure (a) The original problem (or problem (a)) consisting of a coax probe fed patch antenna, and (b) the generalized reciprocal or test problem (or problem (b)).



$\Sigma \to \infty$
$S_a^{\pm}$ on patch
S_p on probe
S_g on COAX GAP
S_c on PEC plane.

Figure Surface $(\Sigma + S_a^+ + S_a^- + S_p + S_g + S_c)$ bounds the external volume V_0.

The direct solution to the original problem (a) above requires one to essentially employ a microstrip Green's function; the latter Green's function is generally expressed in the form of a Sommerfeld type integral over an infinite limit. This approach is complicated.

It is easier in the FAR ZONE case to solve the original problem in terms of an appropriate simpler test problem via a generalized reciprocity/reaction theorem which relates the two problems. The test problem (b) shown above shows a "distant" (FAR ZONE OF PATCH) test source illuminating ONLY a grounded material slab with the same electrical parameters and thickness as in the original problem.

Let the probe fed patch radiate $(\bar{E}, \bar{H})$ in V_0; also let a test source $\bar{J}_t = \hat{u}\,\delta(\bar{r}'' - \bar{r})$ exist at $\bar{r}'' = \bar{r}$, where $\bar{r}$ is a position vector to an observation point P in the FAR ZONE of the patch.

Let the test source generate $(\bar{E}_t, \bar{H}_t)$ in V_0.

From conservation of reactions, $\langle a,b\rangle = \langle b,a\rangle$ in V_0:

$$\int_{V_0} \bar{E}(\bar{r}'')\cdot\bar{J}_t(\bar{r}'')\,dv'' = \int_{S_a^+ + S_a^- + S_p + S_c + S_g} \bar{E}_t(\bar{r}')\cdot\bar{J}_s(\bar{r}')\,ds' - \int_{S_g} \bar{H}_t(\bar{r}')\cdot\bar{M}_s(\bar{r}')\,ds'.$$

It is noted that V_0 is bounded by $(\Sigma + S_a^+ + S_a^- + S_p + S_c + S_g)$. Note that the only source in V_0 is $\bar{J}_t$. Also, $\Sigma \to \infty$ does not contribute to the integrals on the RHS. Furthermore,

$$\bar{J}_s \equiv \hat{n}\times\bar{H} \quad ; \quad \bar{M}_s \equiv \bar{E}\times\hat{n} \; ;$$

$$(\bar{E}\times\bar{H}_t - \bar{E}_t\times\bar{H})\cdot\hat{n} = -\bar{H}_t\cdot(\bar{E}\times\hat{n}) + \bar{E}_t\cdot(\hat{n}\times\bar{H}) = \bar{E}_t\cdot\bar{J}_s - \bar{H}_t\cdot\bar{M}_s\,,$$

have been utilized above for $\langle a,b\rangle = \langle b,a\rangle$.

Also $\hat{n} = \hat{n}_+$ on S_a^+; $\hat{n} = \hat{n}_-$ on S_a^-. Thus

$$\int_{S_a^+ + S_a^-} \bar{E}_t\cdot\bar{J}_s\,ds = \int_{PATCH} \bar{E}_t\cdot\bar{J}_{patch}\,ds\;; \quad \bar{J}_{patch} = \hat{n}_+\times\left[\bar{H}\big|_{S_a^+} - \bar{H}\big|_{S_a^-}\right]$$

$$(\hat{n}_- = -\hat{n}_+).$$

$$\int_{S_p} \bar{E}_t \cdot \bar{J}_s \, ds' = \int_{-d}^{0} dz' \int_{0}^{2\pi} \bar{E}_t(\bar{r}') \cdot \frac{\hat{z} I_o}{2\pi a} (a\, d\psi') \quad ; \quad \bar{J}_s \Big|_{S_p} \approx \frac{I_o}{2\pi a} \hat{z}$$

where I_o is an assumed current at the base of the coax probe; it is assumed to have the same value over the entire probe as long as the probe (of radius = a) is short ($k_m d = \omega\sqrt{\mu_o \epsilon}\, d << 1$) where d = probe length.

Since $\hat{n} \times \bar{E}_t = 0$ on $S_c + S_g$ (see test problem (b) geometry in figure), the contribution $\int_{S_c+S_g} \bar{E}_t \cdot \bar{J}_s \, ds' = 0$.

Also, $\bar{M}_s$ exists only in the gap of the COAX feed. Here $\bar{M}_s = \bar{E}_g(\bar{r}') \times \hat{z}$, where $\bar{E}_g$ is assumed to be the TEM coax modal electric field. $\bar{E}_g = v_o \bar{e}_o$.

Since $\bar{J}_t(\bar{r}'') = \hat{u}\, \delta(\bar{r}'' - \bar{r})$, the relation $\langle a,b \rangle = \langle b,a \rangle$ yields:

$$\bar{E}(\bar{r}) \cdot \hat{u} = \int_{patch} \bar{E}_t(\bar{r}') \cdot \bar{J}_{patch}(\bar{r}') ds' + I_o \int_{-d}^{0} \hat{z} \cdot \bar{E}_t(\bar{r}') dz' + \int_{S_g} \bar{H}_t(\bar{r}') \cdot V_o \bar{e}_o \times \hat{z}\, ds'.$$

One may assume a dominant mode current on the patch:

$$\bar{J}_{patch}(\bar{r}') \approx \hat{y} J_o \cos\frac{\pi}{L} y' \quad ; \quad |y'| \le \frac{L}{2} \, .$$

A connection can be made between J_o and I_o, but it will not be discussed; however, it may be mentioned that it is based on enforcing the Kirchhoff current law where the probe wire attaches to the patch. Also, the v_o and I_o given above may be related by

$$\frac{v_o}{I_o} \approx Z_a \quad \text{(antenna impedance)}.$$

It now remains to find $(\bar{E}_t, \bar{H}_t)$ when P is in the far zone of the PATCH. To determine $(\bar{E}_t, \bar{H}_t)$ in a simple fashion, it is useful to find $\bar{E}^i$ which is the field of $\bar{J}_t$ in free space (without the grounded slab present). Thus,

$$\bar{E}^i(\bar{r}') \approx \frac{jk_0 Z_0}{4\pi}(-\hat{\xi}) \times (-\hat{\xi}) \times \frac{e^{-jk_0\xi}}{\xi}\hat{u} \; ; \; k_0 = \omega\sqrt{\mu_0\epsilon_0} \; ; \; Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}.$$

As $\bar{r}'' \to \bar{r}$ (when P moves to the FAR ZONE of the patch), then $-\hat{\xi} \to \hat{R} \to \hat{r}$. Also, $\xi = |\bar{r}'' - \bar{r}'| \to |\bar{r} - \bar{r}'|$

$$\therefore \quad \xi = |\bar{r}'' - \bar{r}'| \to |\bar{r} - \bar{r}'| \; ; \; |\bar{r} - \bar{r}'| \approx r - \hat{r} \cdot \bar{r}'.$$

$$(\text{and } -\hat{\xi} \to \hat{r})$$

so,

$$\bar{E}^i(\bar{r}') \approx \frac{jk_0 Z_0}{4\pi}(\hat{r} \times \hat{r} \times \hat{u})\frac{e^{-jk_0 r}}{r} \, e^{jk_0 \hat{r} \cdot \bar{r}'}.$$

i.e.,

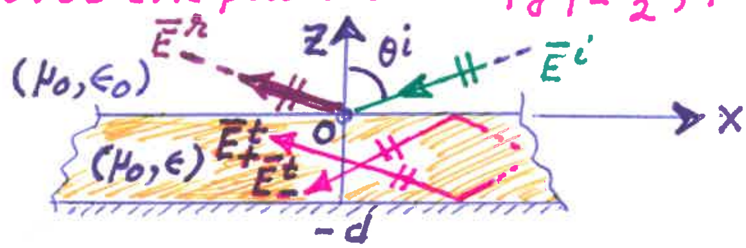
$$\bar{E}^i(\bar{r}') = \bar{A}_0 \, e^{-j\bar{k}^i \cdot \bar{r}'} \; ; \quad \bar{k}^i \equiv k_0(-\hat{r})$$

where

$$A_0 \equiv \frac{-jk_0 Z_0}{4\pi}\left(\hat{r} \times \hat{r} \times \begin{bmatrix}\hat{\theta} \\ \hat{\phi}\end{bmatrix}\right)\frac{e^{-jk_0 r}}{r} \; ; \quad \text{if } \hat{u} \equiv \begin{bmatrix}\hat{\theta} \\ \hat{\phi}\end{bmatrix}$$

$$\therefore \quad \to \quad \bar{E}^i(\bar{r}') = A_0 \begin{bmatrix}\hat{\theta} \\ \hat{\phi}\end{bmatrix} e^{-j\bar{k}^i \cdot \bar{r}'} \Longrightarrow \text{(LOCAL PLANE WAVE)}$$

$\Longrightarrow$ (over the patch area $|y'| \leq \frac{L}{2}$; $|x'| \leq \frac{W}{2}$).



(y=0 plane for convenience)

E_t can be found from the solution to problem of PLANE WAVE reflection and transmission at the grounded material/dielectric slab interface.

$$\bar{E}_t \approx \begin{pmatrix} \bar{E}^i + \bar{E}^r & , \text{ for } z' \geq 0 \\ \bar{E}^t_+ + \bar{E}^t_- & , \text{ for } -d \leq z' < 0 \end{pmatrix}.$$

LET "$y = 0$" FOR CONVENIENCE.

Case A: $\hat{u} = \hat{\theta}$

$$\bar{E}^i(\bar{r}') = A_o \hat{\theta} e^{-j\bar{k}^i \cdot \bar{r}'} = A_o \hat{\theta} e^{-jk_o(-\hat{x}\sin\theta^i - \hat{z}\cos\theta^i)\cdot[x'\hat{x} + y'\hat{y} + z'\hat{z}]}$$

$\therefore$

$$E^i(\bar{r}') = A_o[-\hat{z}\sin\theta^i + \hat{x}\cos\theta^i]e^{jk_o(x'\sin\theta^i + z'\cos\theta^i)}$$

$$\bar{E}^r(\bar{r}') = (R)A_o[-\hat{z}\sin\theta^i - \hat{x}\cos\theta^i]e^{jk_o(x'\sin\theta^i - z'\cos\theta^i)}$$

$$R = \frac{k_m\cos\theta^i - j(Z_m/Z_o)\sqrt{k_m^2 - k_o^2\sin^2\theta^i}\tan\left(\sqrt{k_m^2 - k_o^2\sin^2\theta^i}\, d\right)}{k_m\cos\theta^i + j(Z_m/Z_o)\sqrt{k_m^2 - k_o^2\sin^2\theta^i}\tan\left(\sqrt{k_m^2 - k_o^2\sin^2\theta^i}\, d\right)}$$

$Z_m = \sqrt{\frac{\mu_o}{\epsilon}}$; $k_m = \omega\sqrt{\mu_o\epsilon}$. Also $Y_m = (Z_m)^{-1}$.

$\bar{H}^{i,r} = Y_o\, \hat{k}^{i,r} \times \bar{E}^{i,r}$; $Y_o = (Z_o)^{-1}$; $Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$; $k_o = \omega\sqrt{\mu_o\epsilon_o}$.

Thus $\int_{patch} \bar{E}_t \cdot \bar{J}_{patch}\, ds'$ becomes

$$J_o \int_{-L/2}^{L/2} dy' \int_{-w/2}^{w/2} dx' \cos\frac{\pi y'}{L} \cdot \left(\bar{E}^i + \bar{E}^r\right)\Big|_{z'=0} \cdot \hat{y} = 0$$

$\therefore$ $\langle a, b\rangle = \langle b, a\rangle$ yields:

$\Rightarrow$ $\bar{E}(\bar{r})\cdot\hat{\theta} \cong 0$ (FAR ZONE),

if one assumes the radiation from the tiny probe (at S_p) and tiny gap (at S_g) can also be negligible.

For completeness:

$$\bar{H}^i = \hat{y}\left(-\frac{A_o}{Z_o}\right)e^{jk_o(x'\sin\theta^i + z'\cos\theta^i)},$$

$$\bar{H}^r = \hat{y}R\left(-\frac{A_o}{Z_o}\right)e^{jk_o(x'\sin\theta^i - z'\cos\theta^i)},$$

(NOTE: $z' = 0$ on patch)

LET "$y=0$" FOR CONVENIENCE.

Case B: $\hat{u} = \hat{\phi}$

$$\bar{E}^i(\bar{r}') = A_0\, \hat{\phi}\, e^{-j\bar{k}^i\cdot\bar{r}'} = \hat{y} A_0\, e^{jk_0(x'\sin\theta^i + z'\cos\theta^i)}$$

$$\bar{E}^r(\bar{r}') = \hat{y}(R)A_0\, e^{jk_0(x'\sin\theta^i - z'\cos\theta^i)}$$

$$R = \frac{k_m\cos\theta^i + j\,(Y_m/Y_0)\sqrt{k_m^2 - k_0^2\sin^2\theta^i}\,\tan\left(\sqrt{k_m^2 - k_0^2\sin^2\theta^i}\,d\right)}{k_m\cos\theta^i - j\,(Y_m/Y_0)\sqrt{k_m^2 - k_0^2\sin^2\theta^i}\,\tan\left(\sqrt{k_m^2 - k_0^2\sin^2\theta^i}\,d\right)}$$

$$\bar{E}_t = \bar{E}^i + \bar{E}^r \quad \text{for } z' \geq 0 .$$

Once again, neglecting the contributions from the tiny probe (at S_p) and the tiny gap (at S_g), one obtains via $\langle a, b\rangle = \langle b, a\rangle$ the following:

$$\bar{E}(\bar{r})\cdot\hat{\phi}\Big|_{y=0} = \bar{E}(\bar{r})\cdot\hat{y}\Big|_{y=0} = \frac{-jkZ_0}{4\pi}\cdot\frac{e^{-jk_0 r}}{r}\, J_0\,(1+R)\cdot \mathcal{I} .$$

where

$$\mathcal{I} = \int_{-w/2}^{w/2} dx'\, e^{jk_0 x'\sin\theta^i} \cdot \int_{-L/2}^{L/2} dy' \cos\frac{\pi y'}{L} .$$

The above integrals can be easily evaluated in closed form.

Slotted rectangular waveguide arrays [4]

Consider a rectangular metallic closed waveguide whose cross sectional dimensions are "a" and "b", where "a" is the width, and "b" is the height, respectively.

The waveguide is operated in the dominant TE_{10} mode.

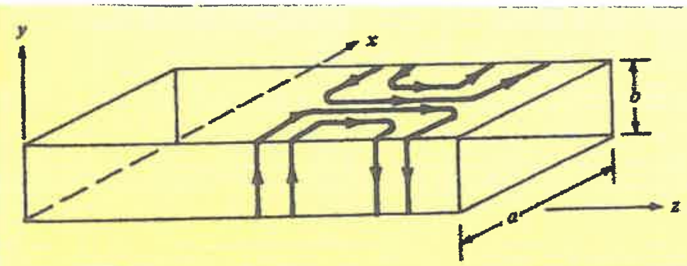


Figure A rectangular waveguide and the surface current for the TE_{10} mode.

The EM fields within the guide for the TE_{10} mode are:

$$\bar{E} = \hat{y}\, E_0 \sin\frac{\pi x}{a}\, e^{-j\beta z} \; ; \; \bar{H} = E_0\left[-\hat{x}\, Y_g \sin\frac{\pi x}{a} + \hat{z}\, j\frac{\pi}{k_0 a} Y_0 \cos\frac{\pi x}{a}\right] e^{-j\beta z} .$$

$$K_0 = \omega\sqrt{\mu_0 \epsilon_0} \; ; \; Y_0 = Z_0^{-1} = \sqrt{\frac{\epsilon_0}{\mu_0}} \; ; \; Y_g = Y_0 \frac{\beta}{k_0} \; ; \; \beta = \sqrt{k_0^2 - \left(\frac{\pi}{a}\right)^2} .$$

Let the wavelength in the guide be λ_g, and let the free space wavelength be λ_0, where,

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi}{\lambda_0} \rightarrow \lambda_0 = \frac{2\pi}{k} \; ; \text{likewise } \lambda_g \equiv \frac{2\pi}{\beta} .$$

$\Rightarrow$ (NOTE : $\beta < k_0$) $\rightarrow$ TE_{10} is a FAST-WAVE.

On the INNER TOP (broad) wall, the surface current $\bar{J}_s = -\hat{y} \times \bar{H}$

$$\therefore \; \bar{J}_s = -\hat{y} \times \bar{H} = \left[-\hat{x}\, j E_0 \frac{\pi Y_0}{k_0 a} \cos\frac{\pi x}{a} - \hat{z}\, E_0 Y_g \sin\frac{\pi x}{a}\right] e^{-j\beta z} . \text{ (see above fig.)}$$

- Slots cut along (or parallel) to current DO NOT RADIATE.
- Slots cut to perturb the current DO RADIATE.

RESONANT BROADSIDE SLOTTED WAVEGUIDE ARRAY [4]

A resonant slotted waveguide array radiates a broadside beam. The slots are spaced $\frac{\lambda_g}{2}$ apart; this requirement makes the array extremely narrow band. The slots are also offset from the centerline and alternates between opposite sides to introduce an additional phase shift of π radians (– this is in addition to $\frac{\lambda_g}{2}$ spacing which provides a phase change of π radians). Thus, ALL SLOTS RADIATE "IN PHASE."

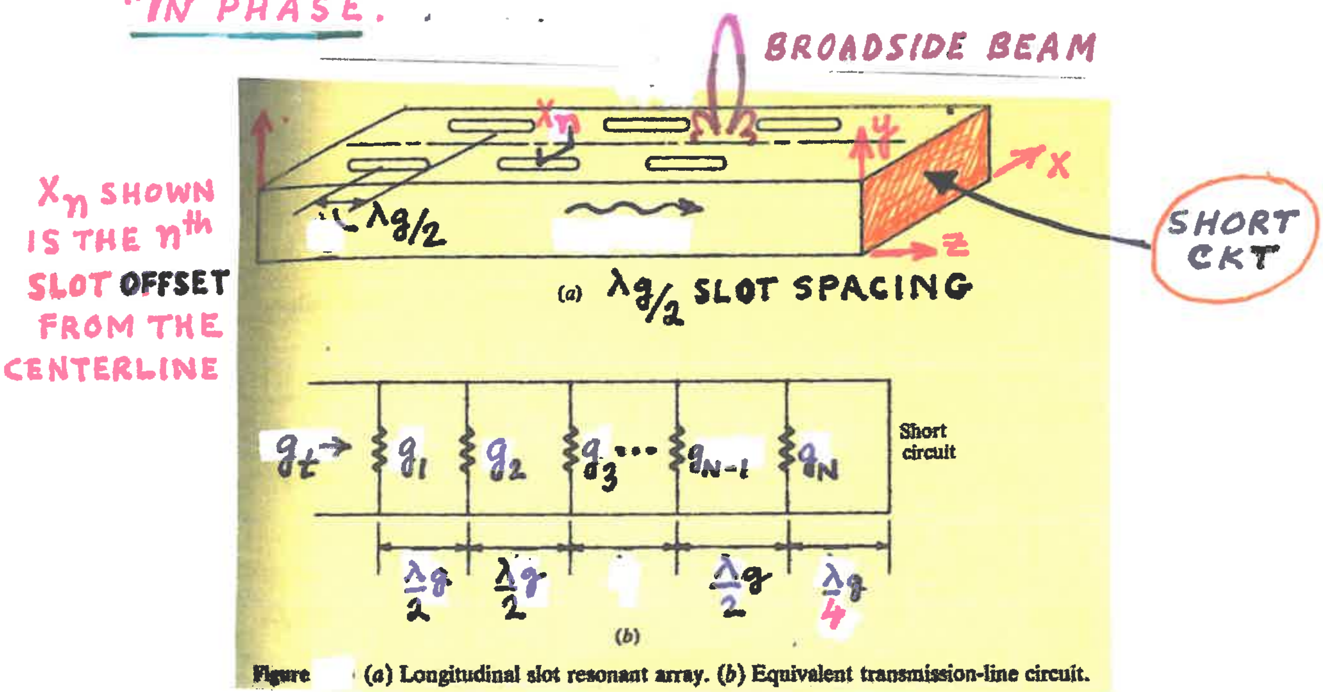


Figure (a) Longitudinal slot resonant array. (b) Equivalent transmission-line circuit.

The array configuration is shown above.

For a thin longitudinal "RESONANT SLOT" at "any" n^{th} slot location, the slot admittance (as seen by the waveguide) is purely REAL (= CONDUCTANCE).

The n^{th} slot conductance, G_n, depends on the offset distance, x_n, from the centerline. The amount of radiation by the n^{th} slot can be controlled by the offset distance, x_n. From [4, 5]:

$$\frac{G_n}{Y_g} = 2.09 \frac{\lambda_g}{\lambda_0} \frac{a}{b} \cos^2\left(\frac{\pi\lambda_0}{2\lambda_g}\right) \cdot \sin^2\left(\frac{\pi x_n}{a}\right) \equiv g_n \leftarrow$$

where g_n is the <u>normalized</u> conductance of the n^{th} slot. The equivalent circuit for the broadwall slot is a SHUNT CONDUCTANCE across a transmission line representing the TE_{10} (dominant) mode. Let the characteristic impedance be <u>normalized</u> so it has a UNIT value, with propagation constant β on the line.

Let a short circuit be introduced at $\frac{\lambda_g}{4}$ from the last slot; this introduces an open circuit in shunt with the last (N^{th}) slot.

The net conductance, g_t, seen at the input is the sum of all N shunt conductances:

$$g_t = g_1 + g_2 + g_3 + \cdots + g_n + \cdots + g_{N-1} + g_N + \overset{\text{SHORT}}{0}$$

$\therefore$

$$g_t = \sum_{n=1}^{N} g_n .$$

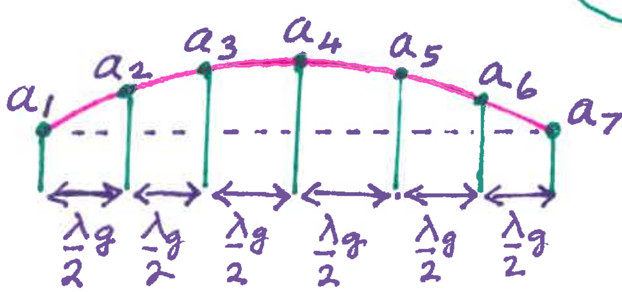
If $g_t = 1$, then no <u>power</u> reflects back to the input side, and hence all the <u>power</u> is radiated. The <u>power</u> radiated by the n^{th} slot is $\frac{1}{2}|V|^2 g_n$, where V is the line voltage. Thus the excitation <u>amplitude</u>, a_n, proportional to $\sqrt{g_n}$.

$$a_n \propto \sqrt{g_n} \Longrightarrow \boxed{g_n = C\, a_n^2}$$

where C is the constant of proportionality.

Since $g_t = \sum_{n=1}^{N} g_n = 1$ (for all the power to be radiated),

$$g_t = 1 = C \sum_{n=1}^{N} a_n^2 \longrightarrow \boxed{C = \frac{1}{\sum_{1}^{N} a_n^2}}.$$



Consider a seven (7) element slot array aperture distribution consisting of a tapered function plus a constant (pedestal) value. For example:

$a_1 = a_7 = 1$; $a_2 = a_6 = 1+3$; $a_3 = a_5 = 1+4$; $a_4 = 1+5$.

① $C = [1+1+16+16+25+25+36]^{-1}$.

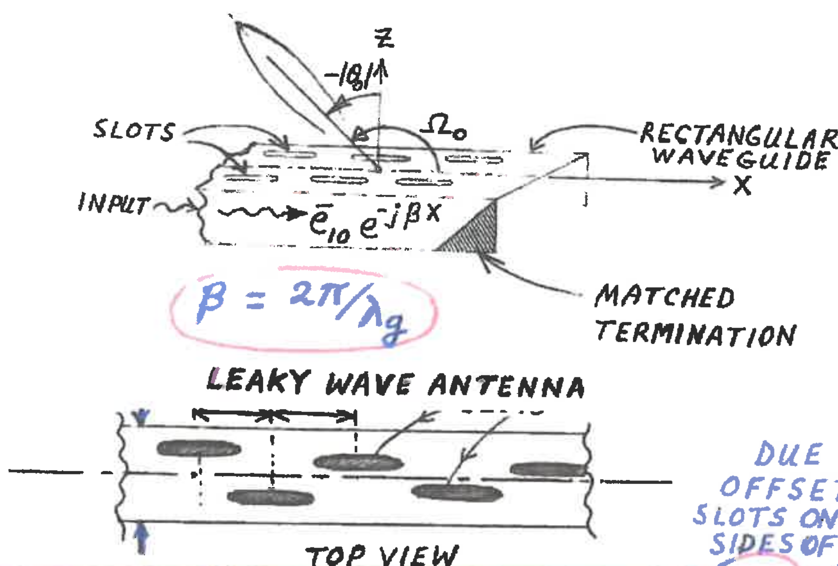
② Find g_n from $g_n = C a_n^2$ for $n = 1$ to $n = N = 7$.

③ Use the value of g_n in ② into the formula presented earlier to find the offsets x_n. Let the operating frequency be 10 GHz with X-band guide having dimensions $a = 0.9''$ and $b = 0.4''$. Note that $\lambda_0 = 3$ cm and $\lambda_g = 3.975$ cm. Also $1'' = 2.54$ cm.

Frequency Scanned Array

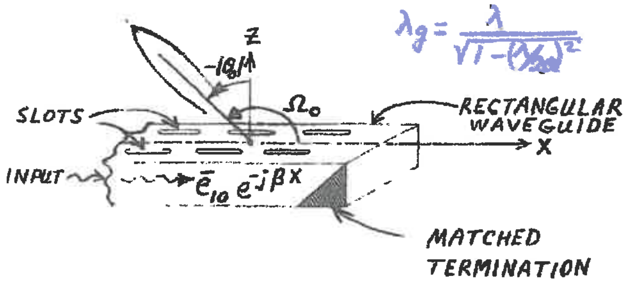
Instead of PHASED SCANNED ARRAYS, one can also have FREQUENCY SCANNED ARRAY

FOR EXAMPLE : A 1-D LEAKY WAVEGUIDE ARRAY [4]



DUE TO OFFSET OF SLOTS ON OPP SIDES OF CENTER

$$\boxed{AF = \sum_{n=1}^{N} a_n\, e^{j(k\sin\theta - \beta)nd}\, e^{jn\pi}}$$



LEAKY WAVE ANTENNA

$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}$

$$\beta = \sqrt{k^2 - \left(\frac{\pi}{2}\right)^2} \quad \text{for} \quad TE_{10} \quad \text{mode}$$

$$k = \frac{2\pi}{\lambda} \quad \beta = \frac{2\pi}{\lambda_g} < k \quad \text{(fast wave)}$$

$$\boxed{AF = \sum_{n=1}^{N} a_n e^{j(k\sin\theta - \beta)nd} e^{jn\pi}}$$

AF peaks occur at

$$n\pi + \left(k\sin\theta_p - \beta\right)nd = 2m\pi \qquad m = 0, \pm 1 \cdots$$

or

$$\boxed{n\pi + \left(k\cos\Omega_p - \beta\right)nd = 2m\pi \qquad p \equiv \frac{m}{n} = 0, \pm 1, \pm 2 \cdots}$$

$$(2p-1)\pi = \left(k\sin\theta_p - \beta\right)d$$

$$\Rightarrow \boxed{d = \frac{(2p-1)\pi}{k\sin\theta_p - \beta} = \frac{(2p-1)\lambda\lambda_g}{2\left(\lambda_g\sin\theta_p - \lambda\right)}}$$